

INTRODUCTION TO HOMOLOGICAL GEOMETRY: PART I

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Given a space M , one may attempt to construct various natural cohomology algebras such as the ordinary (simplicial, singular, etc.) cohomology algebra $H^*(M)$ and the quantum cohomology algebras $QH^*(M)$ and $\widetilde{QH}^*(M)$. For example, if M is the n -dimensional complex projective space $\mathbf{C}P^n$, then

$$H^*(\mathbf{C}P^n) \cong \mathbf{C}[p]/\langle p^{n+1} \rangle, \quad \widetilde{QH}^*(\mathbf{C}P^n) \cong \mathbf{C}[p, q]/\langle p^{n+1} - q \rangle.$$

In the first case the algebraic variety defined by the equation $p^{n+1} = 0$ is not very interesting, but in the second case we have the nontrivial variety $p^{n+1} - q = 0$ in \mathbf{C}^2 . Roughly speaking, “homological geometry” is concerned with “geometry of the algebraic variety V_M , where the algebra of functions on V_M is the quantum cohomology algebra of the space M ”.

Our main sources of inspiration for this subject are the stimulating papers [Gi-Ki] and [Gi1]-[Gi6]. The excellent (if idiosyncratic) survey papers [Au1]-[Au4] amplify and explain some of this material. Our aim in these lectures is very modest: if Audin is an introduction to Givental, then the first part of this survey of homological geometry will be an introduction to Audin.

We shall not discuss the rigorous definition of quantum cohomology. For this, the reader should consult the articles [Ko], [Ru-Ti] and the books and survey articles listed later on. Nor do we discuss the historical motivation, which comes from physics. Nevertheless, we shall begin by giving an informal introduction to quantum cohomology, together with some very explicit calculations in the appendices. Readers who have found quantum cohomology intimidating may wonder how it will be possible to contemplate applications of the theory, after such a superficial treatment of the foundations. My answer would be that this is already standard practice for *ordinary* cohomology theory; the rigorous foundations of cohomology theory are unavoidably messy, yet there is no difficulty in computing (for example) the cohomology rings for simple spaces such as surfaces or projective spaces. In

quantum cohomology we face the same situation. We define a product operation by intersecting certain special kinds of cycles, and, even though the general definition is complicated, we can perform the calculations satisfactorily for nice spaces (such as homogeneous Kähler manifolds).

Our main purpose is to describe a path from this naive intersection-theoretic formulation of quantum cohomology to the way in which differential equations — especially those related to differential geometry and the theory of integrable systems — enter into quantum cohomology. We begin, therefore, with some generalities on flat connections and Frobenius manifolds, and then discuss quantum cohomology as an example from this point of view. In sections 4, 5 and 6 we give three examples of how differential equations arise. In the last section we indicate very briefly how the flat connection defined by quantum cohomology underlies these phenomena. In part II we shall consider the differential equations in more depth.

In addition to the articles mentioned earlier, the books and survey articles [BCPP], [Co-Ka], [Du], [Ma], [Mc-Sa], [Si], [Ti], [Vo] may be consulted for further information and many more references. Although we shall not reach the point of discussing any of the famous enumerative results (because we use only the “small” quantum product) or mirror symmetry, we hope that the reader will be prepared to go on to these topics after reading this introduction.

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§1 FLAT CONNECTIONS AND FROBENIUS ALGEBRAS

References: [Du], [Mc-Sa]

Let W be a finite-dimensional vector space (real in this section, but complex in future sections). Let d denote the directional derivative operator on functions $W \rightarrow W$; thus, for $X \in W$ and $Y : W \rightarrow W$, we denote by $d_X Y$ the derivative of Y in the direction of X . We regard d as the standard flat connection (covariant derivative operator) on the manifold W .

Any other connection on W is of the form $\nabla = d + \omega$, where ω is a 1-form on W with values in the Lie algebra $\text{End}(W)$. Given any such ω , we define an associated family of connections ∇^λ by $\nabla^\lambda = d + \lambda\omega$, where $\lambda \in \mathbf{R}$.

It is easy to check that

$$(1) \quad \begin{aligned} \nabla \text{ is flat} &\iff d\omega + \omega \wedge \omega = 0 \\ \nabla^\lambda \text{ is flat for all } \lambda &\iff d\omega = 0, \omega \wedge \omega = 0 \end{aligned}$$

and

$$(2) \quad \begin{aligned} \nabla \text{ has zero torsion} &\iff \omega(X)Y = \omega(Y)X \text{ for all } X, Y \\ &\iff \nabla^\lambda \text{ has zero torsion for all } \lambda. \end{aligned}$$

Similarly, if $(\ , \)$ is a symmetric bilinear form on W , then

$$(3) \quad \begin{aligned} \nabla \text{ is compatible with } (\ , \) &\iff (\omega(X)Y, Z) + (Y, \omega(X)Z) = 0 \text{ for all } X, Y, Z \\ &\iff \nabla^\lambda \text{ is compatible with } (\ , \) \text{ for all } \lambda. \end{aligned}$$

We shall be interested in the special case where W is a *Frobenius algebra*. This means that W has a commutative associative product operation \circ , so that W becomes an algebra, and that the bilinear form is nondegenerate and also satisfies the “Frobenius condition” $(X \circ Y, Z) = (X, Y \circ Z)$ for all $X, Y, Z \in W$.

For a Frobenius algebra W , we shall consider the 1-form $\omega(X)Y = X \circ Y$, and the corresponding connections

$$\nabla_X^\lambda Y = d_X Y + \lambda X \circ Y.$$

Proposition. *For any Frobenius algebra we have*

- (i) ∇^λ is flat for all λ
- (ii) ∇^λ has zero torsion for all λ

Proof. We use conditions (1) and (2) above. Commutativity of \circ gives (ii). For (i), the fact that $\omega_t(X)(Y)$ is independent of $t \in W$ gives $d\omega = 0$, and commutativity and associativity of \circ gives $\omega \wedge \omega = 0$.

The connection ∇^λ is compatible with the bilinear form $(\ , \)$ if and only if $\lambda = 0$, i.e. $\nabla^\lambda = d$. Indeed, the Frobenius condition may be interpreted as a kind of “skew-compatibility”.

More generally, one can consider a family \circ_t of Frobenius algebra structures on W . We shall be interested in the case where the family is parametrized by $t \in W$. Since the tangent bundle of the manifold W is trivial, we may then regard \circ_t as a Frobenius algebra structure on the tangent space $T_t W$ at t . The formula

$$\nabla_X^\lambda Y = d_X Y + \lambda X \circ_t Y$$

defines a connection on the manifold W ; it is called the *Dubrovin connection*.

From the definition of Frobenius structure it can be shown (as in the proposition) that ∇^λ has zero torsion, and also that $\omega \wedge \omega = 0$, where $\omega_t(X)(Y) = X \circ_t Y$. However, it is not necessarily flat, because we do not necessarily have $d\omega = 0$.

We are now in a situation similar to that of having an almost complex structure on a manifold, i.e. where there is a complex structure on each tangent space, but these complex structures are not necessarily “integrable”. The manifold W , with the family of Frobenius structures $\{\circ_t \mid t \in W\}$, is in fact an example of a pre-Frobenius manifold (see [Du]). It will be a Frobenius manifold — i.e. it will be regarded as “integrable” — if the Dubrovin connection is flat for all λ , i.e. if $d\omega = 0$.

§2 AN EXAMPLE OF A FROBENIUS ALGEBRA: $(H^*(M), \circ_t)$

Reference: [Mc-Sa]

Let M be a simply connected (and compact, connected) Kähler manifold, of complex dimension n . We shall assume that the integral cohomology of M is even dimensional and torsion-free, i.e. that

$$(A1) \quad H^*(M; \mathbf{Z}) = \bigoplus_{i=0}^n H^{2i}(M; \mathbf{Z}) \cong \bigoplus_{i=0}^n \mathbf{Z}^{m_{2i}}$$

where $m_{2i} = \text{rank } H^{2i}(M; \mathbf{Z})$.

The role of the vector space of §1 will be played by $W = H^*(M; \mathbf{C}) \cong \bigoplus_{i=0}^n \mathbf{C}^{m_{2i}}$. There is a family $\{\circ_t \mid t \in W\}$ of Frobenius structures on the vector space W , called the *quantum product*. In this section we shall define \circ_t in terms of certain *Gromov-Witten invariants* $\langle A|B|C \rangle_D$. In the next section we shall define $\langle A|B|C \rangle_D$ and give some examples.

We need some notation from ordinary cohomology theory. Let

$$\begin{aligned} \text{PD} : H^i(M; \mathbf{Z}) &\rightarrow H_{2n-i}(M; \mathbf{Z}) \\ &4 \end{aligned}$$

be the Poincaré duality isomorphism. This may be defined as the map which sends a cohomology class x to the “cap product” of x with the fundamental class of M . As far as possible we shall use the notation

$$a, b, c, \dots \in H^*(M; \mathbf{Z}) \quad (\subseteq H^*(M; \mathbf{C}))$$

for cohomology classes, and we write $|a|, |b|, |c|, \dots$ for the degrees of a, b, c, \dots . We shall write

$$A = \text{PD}(a), B = \text{PD}(b), C = \text{PD}(c), \dots \in H_*(M; \mathbf{Z}) \quad (\subseteq H_*(M; \mathbf{C}))$$

for the Poincaré dual homology classes, and $|A|, |B|, |C|, \dots$ for their degrees.

Let

$$\langle , \rangle : H^i(M; \mathbf{Z}) \times H_i(M; \mathbf{Z}) \rightarrow \mathbf{Z}$$

denote the Kronecker (or “evaluation”) pairing; we use the same notation for the extended pairing

$$\langle , \rangle : H^*(M; \mathbf{Z}) \times H_*(M; \mathbf{Z}) \rightarrow \mathbf{Z}$$

(thus, $\langle a, B \rangle$ is zero whenever $|a| \neq |B|$). Since there is no torsion, both these pairings are nondegenerate.

The role of the bilinear form of §1 will be played by the (\mathbf{C} -linear extension of the) “intersection pairing”, which is defined by

$$(,) : H^*(M; \mathbf{Z}) \times H^*(M; \mathbf{Z}) \rightarrow \mathbf{Z}, \quad (a, b) = \langle ab, M \rangle.$$

On the right hand side, M denotes the fundamental homology class of the manifold M . It is an element of $H_{2n}(M; \mathbf{Z})$, and its Poincaré dual cohomology class — the identity element of the cohomology algebra — will be denoted by $1 \in H^0(M; \mathbf{Z})$ (an exception to our notational convention for cohomology classes!). The homology class represented by a single point of M will be denoted by $Z \in H_0(M; \mathbf{Z})$, and its Poincaré dual cohomology class will be denoted in accordance with our convention by z .

Now, the well known duality between the cup product and the Kronecker pairing may be expressed by the formula

$$\langle ab, M \rangle = \langle a, B \rangle = \langle b, A \rangle.$$

From the nondegeneracy of the Kronecker pairing (in this situation), it follows that the intersection pairing $(,)$ is nondegenerate. The formula also shows that the cup product satisfies the Frobenius condition $(ab, c) = (a, bc)$, because both sides of this equation

are equal to $\langle abc, M \rangle$. Hence the cup product and the intersection pairing give rise to a Frobenius algebra structure on the vector space W . It should be noted that the intersection pairing is always indefinite (unless M is zero-dimensional).

The resulting Frobenius structure on the manifold W is trivial, in the sense that the associated 1-form $\omega_t(x)(y) = xy$ is constant, i.e. independent of t . The condition $d\omega = 0$ is therefore automatically satisfied. On the other hand, the quantum product will give a nontrivial example, as we shall see next.

The cup product on $H^*(M; \mathbf{Z})$ may be specified in terms of its “structure constants”. To do this, we choose generators as follows:

$$H_*(M; \mathbf{Z}) = \bigoplus_{i=0}^s \mathbf{Z} A_i$$

$$H^*(M; \mathbf{Z}) = \bigoplus_{i=0}^s \mathbf{Z} a_i$$

and we define “Kronecker dual” cohomology classes a_0^c, \dots, a_s^c (i.e. the dual basis with respect to $(,)$) by $a_i a_j^c = \delta_{ij} z$. Then for any i, j we have

$$a_i a_j = \sum_{\{\alpha \mid |a_\alpha| = |a_i| + |a_j|\}} \lambda_\alpha^{i,j} a_\alpha^c$$

for some $\lambda_1^{i,j}, \dots, \lambda_s^{i,j} \in \mathbf{Z}$. These structure constants are given by

$$\lambda_k^{i,j} = \langle a_i a_j a_k, M \rangle.$$

Observe that if $\lambda_k^{i,j} \neq 0$ then the numerical condition

$$|a_i| + |a_j| + |a_k| = 2n$$

must be satisfied.

Definition. For cohomology classes a, b, c we define $\langle A|B|C \rangle_0 = \langle ab, C \rangle = \langle abc, M \rangle$.

Giving all the structure constants is the same thing as giving all “triple products” $\langle A|B|C \rangle_0$.

The quantum product will be determined by a larger family of triple products denoted by $\langle A|B|C \rangle_D$, where D varies in $H_2(M; \mathbf{Z})$. (These are the Gromov-Witten invariants

$\psi_{3,0}^D(A \otimes B \otimes C)$.) We shall define these new triple products in the next section; for the rest of this section we shall just assume that

(A2) $\langle A|B|C\rangle_D \in \mathbf{Z}$ is defined for any $A, B, C \in H_*(M; \mathbf{Z})$, $D \in H_2(M; \mathbf{Z})$. Moreover, $\langle A|B|C\rangle_D \in \mathbf{Z}$ is linear and symmetrical in A, B, C .

To define $a \circ_t b$ for $a, b \in W$ and $t \in W$, it suffices to define $\langle a \circ_t b, C \rangle$ for all $C \in H_*(M; \mathbf{C})$. We shall give a definition only for $t \in H^2(M; \mathbf{C})$; the family $\{\circ_t \mid t \in H^2(M; \mathbf{C})\}$ is called the “small quantum product”. The definition is:

Definition. $\langle a \circ_t b, C \rangle = \sum_{D \in H_2(M; \mathbf{Z})} \langle A|B|C\rangle_D e^{\langle t, D \rangle}$.

We assume that

(A3) in the above definition, the sum on the right hand side is finite.

Observe that as “ $t \rightarrow -\infty$ ” the right hand side converges to $\langle A|B|C\rangle_0$; hence $a \circ_t b$ converges to the cup product ab . In this sense, the quantum product is a deformation of the cup product.

The main result concerning this quantum product is:

Theorem. $\{\circ_t \mid t \in W\}$ defines a Frobenius structure on $W = H^*(M; \mathbf{C})$.

The proof has two nontrivial ingredients: the associativity of the quantum product (which we shall take for granted), and the condition $d\omega = 0$ (which we shall discuss in §7 in the case $t \in H^2(M; \mathbf{C})$).

There is a modification of the quantum product, which is a product operation

$$\circ : H^*(M; \mathbf{C}) \times H^*(M; \mathbf{C}) \rightarrow H^*(M; \mathbf{C}) \otimes \Lambda$$

where Λ is the group algebra $\mathbf{C}[H_2(M; \mathbf{Z})]$. Formally, an element of Λ is a finite sum $\sum_i \lambda_i q^{D_i}$, where $\lambda_i \in \mathbf{C}$, $D_i \in H_2(M; \mathbf{Z})$, and where the symbols q^D are multiplied in the obvious way, i.e. $q^D q^E = q^{D+E}$. (For the time being, therefore, the symbol q has special status as a “formal variable” — it is certainly not a cohomology class!) The definition is:

Definition. $a \circ b = \sum_{D \in H_2(M; \mathbf{Z})} (a \circ b)_D q^D$, where $(a \circ b)_D$ is defined by $\langle (a \circ b)_D, C \rangle = \langle A|B|C\rangle_D$ for all $C \in H_*(M; \mathbf{C})$.

We shall also refer to this operation as the quantum product. By our assumption (A3), the sum on the right hand side of the definition of $a \circ b$ is finite.

Extending the (small) quantum product in a Λ -linear fashion, we can make $W \otimes \Lambda$ into an algebra. This is called the *(small) quantum cohomology algebra* and it is denoted by $QH^*(M; \mathbf{C})$. The Frobenius algebra (W, \circ_t) may be obtained from the quantum cohomology algebra $QH^*(M; \mathbf{C})$ by “putting $q^D = e^{\langle t, D \rangle}$ ”. Thus, (W, \circ_t) and $QH^*(M; \mathbf{C})$ contain essentially the same information.

It is convenient to define a grading on the algebra $W \otimes \Lambda$ by defining

$$|aq^D| = |a| + 2\langle c_1 TM, D \rangle.$$

We shall assume that the quantum products preserve this grading, i.e. that $|a \circ b| = |a| + |b|$. It follows that $|(a \circ b)_D| = |a| + |b| - 2\langle c_1 TM, D \rangle$. Hence, if $\langle A|B|C \rangle_D \neq 0$, then the numerical condition $|a| + |b| - 2\langle c_1 TM, D \rangle = |C|$ must be satisfied, i.e.

$$|a| + |b| + |c| = 2n + 2\langle c_1 TM, D \rangle.$$

The geometrical meaning of this numerical condition will become clear in the next section. We write $QH^i(M; \mathbf{C}) = \{x \in QH^*(M; \mathbf{C}) \mid |x| = i\}$.

One further piece of notation will be useful. In all of our examples we shall choose an identification $H_2(M; \mathbf{Z}) \cong \mathbf{Z}^r$ (for some $r \geq 1$). Having made this choice, we write $D = (s_1, \dots, s_r)$, and $q^D = q_1^{s_1} \dots q_r^{s_r}$. In our examples — although not necessarily for more general manifolds M — it will turn out that the subset $\widetilde{QH}^*(M; \mathbf{C}) = H^*(M; \mathbf{C}) \otimes \mathbf{C}[q_1, \dots, q_r]$ is actually a *subalgebra* of $QH^*(M; \mathbf{C})$ (with respect to the quantum product). This implies in particular that $QH^*(M; \mathbf{C})$ and $\widetilde{QH}^*(M; \mathbf{C})$ contain the same information, and so we can restrict attention to $\widetilde{QH}^*(M; \mathbf{C})$.

§3 EXAMPLES OF $(H^*(M), \circ_t)$ AND $QH^*(M)$

References: [Gi-Ki], [Mc-Sa], [Fu-Pa]

Quantum cohomology depends on the triple products $\langle A|B|C \rangle_D$, and we cannot postpone their definition any longer. In this section we shall give the “naive” definition of $\langle A|B|C \rangle_D$ (as in [Gi-Ki], for example); this has the advantage of simplicity, but the disadvantage that it will be impossible to prove any general theorems. Our main objective is simply to describe some concrete examples as preparation for homological geometry. In this section we shall concentrate on the example $M = \mathbf{C}P^n$, but later on we consider

$M = Gr_k(\mathbf{C}^n)$ (the Grassmannian of complex k -planes in \mathbf{C}^n — in §4), $M = F_n$ (the space of complete flags in \mathbf{C}^n — in Appendix 1), and $M = \Sigma_k$ (the Hirzebruch surface — in Appendix 2).

It has been said that quantum cohomology is “easy to calculate, but hard to define”. This is not as strange as it sounds, because exactly the same can be said for *ordinary* cohomology, if one restricts attention to a few nice spaces. In this spirit, we shall begin by reviewing the definition of the triple product $\langle A|B|C\rangle_0 = \langle ab, C \rangle = \langle abc, M \rangle$ in ordinary cohomology.

The naive definition is

$$\langle A|B|C\rangle_0 = |\tilde{A} \cap \tilde{B} \cap \tilde{C}|,$$

where the right hand side means the number of points (counted with multiplicity) in the intersection $\tilde{A} \cap \tilde{B} \cap \tilde{C}$, where $\tilde{A}, \tilde{B}, \tilde{C}$ are suitable representatives of the homology classes A, B, C . In certain situations this definition is “correct”, in the sense that it gives the usual triple product $\langle abc, M \rangle$. (Similarly, $\langle ab, M \rangle$ may be defined naively as $|\tilde{A} \cap \tilde{B}|$.) For example, in the complex algebraic category, the definition is correct whenever there exist representative algebraic subvarieties $\tilde{A}, \tilde{B}, \tilde{C}$ whose intersection is finite (or empty) — see the appendix of [Fu]. Thus, whenever we are lucky enough to find such representatives, we can calculate $\langle A|B|C\rangle_0$.

The most famous example where this method works is the case $M = Gr_k(\mathbf{C}^n)$ (Schubert calculus). Here all the generators of the homology groups are representable by algebraic cycles (Schubert varieties), and for any three such generators a, b, c satisfying the condition $|a| + |b| + |c| = \dim M$ there exist representatives whose intersection is finite (or empty).

Example: $M = \mathbf{C}P^n$.

We have $H^{2i}(\mathbf{C}P^n; \mathbf{Z}) \cong \mathbf{Z}x_i$ ($0 \leq i \leq n$), where the Poincaré dual homology generator X_i (of degree $2n - 2i$) can be represented by $\tilde{X}_i = \mathbf{P}(V)$, for any complex linear subspace $V \subseteq \mathbf{C}^{n+1}$ of codimension i . Following our usual notational conventions, we shall write $x_0 = 1$ and $x_n = z$.

If $i + j > n$, then $x_i x_j = 0$, since $H^{i+j}(\mathbf{C}P^n; \mathbf{Z}) = 0$. If $i + j \leq n$, then $x_i x_j = \lambda x_{i+j}$ for some λ , and we have to calculate $\lambda = \langle x_i x_j x_{i+j}^c, \mathbf{C}P^n \rangle = \langle X_i | X_j | X_{i+j}^c \rangle_0$.

First, we have $x_i x_{n-j} = \delta_{ij} z$, as there exist linear subspaces V, W of \mathbf{C}^{n+1} of codimensions $i, n - j$ such that $\mathbf{P}(V) \cap \mathbf{P}(W)$ is finite and nonempty if and only if $i = j$, and in this case the intersection consists of a single point (of multiplicity one). This shows that $x_i^c = x_{n-i}$.

To calculate $\langle X_i | X_j | X_{i+j}^c \rangle_0 = \langle X_i | X_j | X_{n-(i+j)} \rangle_0$ (for $i + j \leq n$), we represent the three classes by linear subspaces of \mathbf{C}^{n+1} of codimensions $i, j, n - (i + j)$. If the subspaces are in general position, the codimension of the intersection is $i + j + n - (i + j) = n$, so this triple intersection is a line, and $\tilde{X}_i \cap \tilde{X}_j \cap \tilde{X}_{n-(i+j)}$ is a single point of \mathbf{CP}^n . Since we are taking intersections of linear subspaces, the multiplicity of this point is one. We conclude that $\lambda = 1$, and so $x_i x_j = x_{i+j}$.

The cohomology algebra of \mathbf{CP}^n is therefore isomorphic to $\mathbf{C}[p]/\langle p^{n+1} \rangle$ (where $p = x_1$), as stated in the introduction.

We wish to define $\langle A|B|C \rangle_D$ as a certain intersection number. From our discussion in the last section, it should have the properties

- (a) $D = 0 \implies \langle A|B|C \rangle_D = \langle abc, M \rangle$
- (b) $\langle A|B|C \rangle_D \neq 0 \implies |a| + |b| + |c| = 2n + 2\langle c_1 TM, D \rangle$.

(Actually, (a) will be obvious from the definition, but (b) will have the status of an assumption.)

The naive definition is

$$\langle A|B|C \rangle_D = |\text{Hol}_D^{\tilde{A},p} \cap \text{Hol}_D^{\tilde{B},q} \cap \text{Hol}_D^{\tilde{C},r}|$$

where

$$\text{Hol}_D^{\tilde{A},p} = \{\text{holomorphic maps } f : \mathbf{CP}^1 \rightarrow M \mid f(p) \in \tilde{A} \text{ and } [f] = D\}$$

and where \tilde{A} is a representative of the homology class A . The points p, q, r are three distinct basepoints in \mathbf{CP}^1 . The notation $[f]$ denotes the homotopy class of f , which is an element of $\pi_2(M) \cong H_2(M; \mathbf{Z})$.

We *assume* that there exist $\tilde{A}, \tilde{B}, \tilde{C}$ such that the above intersection is finite (or empty); this is essentially the meaning of assumption (A2) of the previous section. More fundamentally, we shall make the following three assumptions:

(A2a) $\text{Hol}_D^{M,p}$ is a (smooth) complex manifold, of complex dimension $n + \langle c_1 TM, D \rangle$. This will be denoted more briefly by Hol_D .

(A2b) $\text{Hol}_D^{\tilde{A},p}$ is a complex submanifold (or subvariety) of $\text{Hol}_D^{M,p}$, and the complex codimension of $\text{Hol}_D^{\tilde{A},p}$ in $\text{Hol}_D^{M,p}$ is equal to the complex codimension of \tilde{A} in M .

(A2c) If $\tilde{A}, \tilde{B}, \tilde{C}$ intersect transversely, then so do $\text{Hol}_D^{\tilde{A}, p}, \text{Hol}_D^{\tilde{B}, q}, \text{Hol}_D^{\tilde{C}, r}$.

We shall not discuss the extent to which (A2) is equivalent to (A2a), (A2b), (A2c), nor the extent to which these conditions are true. (Certainly all four assumptions hold for homogeneous Kähler manifolds.) However, a brief comment on the origin of the numerical expressions in (A2a) and (A2b) may be helpful. First, the tangent space at $f \in \text{Hol}_D^{M, p}$ — assuming that $\text{Hol}_D^{M, p}$ is a manifold — may be identified with the space of holomorphic sections of the bundle f^*TM . By the Riemann-Roch theorem, the complex dimension of this vector space is $n + \langle c_1 TM, D \rangle$, as stated in (A2a). Regarding (A2b), this would follow from the commutative diagram

$$\begin{array}{ccc} \text{Hol}_D^{\tilde{A}, p} & \xrightarrow{\subseteq} & \text{Hol}_D^{M, p} \\ \downarrow & & \downarrow \\ \tilde{A} & \xrightarrow{\subseteq} & M \end{array}$$

in which the vertical maps are given by $f \mapsto f(p)$, whenever the evaluation map $\text{Hol}_D^{M, p} \rightarrow M$ is a locally trivial fibre bundle (as it will be when M is homogeneous, for example).

It is important to bear in mind that the space Hol_D is finite-dimensional and algebraic, so there is no question of infinite-dimensional analysis here. The technical problems in giving a rigorous definition arise from the noncompactness of Hol_D , and the fact that the transversality condition may not be satisfied. At the end of this section we shall comment briefly on the kind of technical conditions that are needed.

To understand the definition better, let us consider the case where $\pi_2(M) \cong \mathbf{Z}$, e.g. $M = \mathbf{CP}^n$ or $Gr_k(\mathbf{C}^n)$. Then we may write $D = sD_0$ where $s \in \mathbf{Z}$ and D_0 is a generator of $H_2(M; \mathbf{Z})$. For a map $f : \mathbf{CP}^1 \rightarrow M$, we have $[f] = sD_0$ for some s , and we consider this s to be the “degree” of f .

Let $\langle c_1 TM, D_0 \rangle = N$. We shall assume here that $N > 0$ (we have $N = n + 1$ for \mathbf{CP}^n and $N = n$ for $Gr_k(\mathbf{C}^n)$). Then if $\langle A|B|C \rangle_s$, i.e. $\langle A|B|C \rangle_D$, is nonzero, we have the numerical condition

$$|a| + |b| + |c| = 2n + 2sN.$$

If $c_1 TM$ is representable by a Kähler 2-form (as in the case $M = \mathbf{CP}^n$ or $Gr_k(\mathbf{C}^n)$, for example) then the degree of any holomorphic map is nonnegative, so it suffices to consider $s = 0, 1, 2, \dots$. Since $N > 0$, only a finite number of triple products $\langle A|B|C \rangle_s$ can be nonzero, so the series defining \circ and \circ_t are indeed finite series. Let us examine briefly the cases $s = 0, 1, 2$. For $s = 0$, we have¹ $\text{Hol}_D^{\tilde{A}, p} = \tilde{A}$, so the definition of $\langle A|B|C \rangle_s$ reduces to

¹The following three diagrams, and the one in §4, can be downloaded from <http://www.comp.metro-u.ac.jp/~martin>

the definition of $\langle A|B|C \rangle_0$, as it should. We are simply counting the points of the triple intersection $\tilde{A} \cap \tilde{B} \cap \tilde{C}$.

For $s = 1$, the triple product $\langle A|B|C \rangle_1$ counts the holomorphic maps f of degree 1 such that $f(p) \in \tilde{A}$, $f(q) \in \tilde{B}$, $f(r) \in \tilde{C}$. In this case, $|a| + |b| + |c| = 2n + 2N$, so $|a| + |b| + |c| > 2n$, and the triple intersection $\tilde{A} \cap \tilde{B} \cap \tilde{C}$ is “in general” empty.

For $s = 2$, the triple product $\langle A|B|C \rangle_2$ counts the holomorphic maps f of degree 2 such that $f(p) \in \tilde{A}$, $f(q) \in \tilde{B}$, $f(r) \in \tilde{C}$.

Example: $M = \mathbf{CP}^n$.

In this case $N = n + 1$, so the numerical condition (for $\langle A|B|C \rangle_s \neq 0$) is $|a| + |b| + |c| = 2n + 2s(n + 1)$. Since $0 \leq |a|, |b|, |c| \leq 2n$, it follows immediately that $s = 0$ and $s = 1$ are the only relevant values, i.e. that $\langle A|B|C \rangle_s = 0$ for $s \neq 0, 1$.

For $s = 0$ we already know that

$$\langle X_i | X_j | X_k \rangle_0 = \begin{cases} 1 & \text{if } i + j + k = n \\ 0 & \text{otherwise} \end{cases}$$

This is just the triple product for ordinary cohomology.

For $s = 1$ we claim that

$$\langle X_i | X_j | X_k \rangle_1 = \begin{cases} 1 & \text{if } i + j + k = 2n + 1 \\ 0 & \text{otherwise} \end{cases}$$

To prove this, one shows that, when $i + j + k = 2n + 1$, there exist complex linear subspaces E^i, E^j, E^k of \mathbf{C}^{n+1} , of codimensions i, j, k , with the following property: there exist unique complex lines L', L'', L''' such that $L' \subseteq E^i$, $L'' \subseteq E^j$, $L''' \subseteq E^k$ and such that L', L'', L''' span a subspace E of dimension 2. The holomorphic map of degree 1 defined by the inclusion $\mathbf{P}(E) \subseteq \mathbf{C}P^{n+1}$ is then the unique point of the triple intersection $\text{Hol}_1^{\mathbf{P}(E^i), p} \cap \text{Hol}_1^{\mathbf{P}(E^j), q} \cap \text{Hol}_1^{\mathbf{P}(E^k), r}$, and we obtain $\langle X_i | X_j | X_k \rangle_1 = 1$. (The multiplicity of the intersection point is 1 because $\text{Hol}_1^{\mathbf{P}(E^i), p}$ is a smooth subvariety of the space of all holomorphic maps.)

We shall not give a direct proof of the claim, because — if one makes use of the associativity of quantum multiplication — an indirect proof is much easier, as we shall explain later on. This phenomenon is typical: a direct (but tedious) problem of “enumerating” rational curves with certain properties can sometimes be replaced by a simple argument making use of the (nontrivial) properties of quantum cohomology. Spectacular results of this type have been obtained by using more general Gromov-Witten invariants, namely the ones mentioned in the footnote at the beginning of §7.

A simple heuristic argument for the calculation of $\langle X_i | X_j | X_k \rangle_1$ can be made as follows. Let V be the linear subspace of \mathbf{C}^{n+1} of codimension i which is given by the conditions $z_0 = \dots = z_{i-1} = 0$. The space

$$\text{Hol}_s^{\mathbf{P}(V), p} = \{f = [p_0; \dots; p_n] \mid [f] = s \text{ and } p_0(p) = \dots = p_{i-1}(p) = 0\}$$

is a dense open subset of a linear subspace of $\mathbf{P}(\mathbf{C}^{s+1} \times \dots \times \mathbf{C}^{s+1}) \cong \mathbf{P}(\mathbf{C}^{(n+1)(s+1)})$ of codimension i . (The component functions p_i are complex polynomials of degree at most s , such that no point of $\mathbf{C}P^1 = \mathbf{C} \cup \infty$ is a common root of p_0, \dots, p_n .) In the computation of $\langle X_i | X_j | X_k \rangle_1$ we are therefore considering the intersection of (dense open subsets of) three linear subspaces of the $(2n+1)$ -dimensional projective space $\mathbf{P}(\mathbf{C}^{2(n+1)})$ of codimensions i, j, k . We expect a single intersection point if and only if $i + j + k = 2n + 1$, at least if the three subspaces are in general position and if the intersection point can be assumed to

lie in each of the respective open sets. Thus the computation of $\langle X_i | X_j | X_k \rangle_1$ is, like the computation of $\langle X_i | X_j | X_k \rangle_0$, a problem of finding intersections of linear subspaces — the difficulty being the fact that we are dealing with dense open subspaces, rather than the linear subspaces themselves.

Let us calculate the quantum products $x_i \circ x_j$. We have

$$x_i \circ x_j = (x_i \circ x_j)_0 + (x_i \circ x_j)_1 q$$

where $|(x_i \circ x_j)_0| = 2i + 2j$, $|(x_i \circ x_j)_1| = 2i + 2j - 2(n+1)$, and

$$\begin{aligned} \langle (x_i \circ x_j)_0, X_k \rangle &= \langle X_i | X_j | X_k \rangle_0 = \begin{cases} 1 & \text{if } i + j + k = n \\ 0 & \text{otherwise} \end{cases} \\ \langle (x_i \circ x_j)_1, X_k \rangle &= \langle X_i | X_j | X_k \rangle_1 = \begin{cases} 1 & \text{if } i + j + k = 2n + 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

It follows that

$$\begin{aligned} (x_i \circ x_j)_0 &= x_{i+j} & \text{if } 0 \leq i + j \leq n \\ (x_i \circ x_j)_1 &= x_{i+j-(n+1)} & \text{if } n + 1 \leq i + j \leq 2n \end{aligned}$$

and we conclude that

$$x_i \circ x_j = \begin{cases} x_{i+j} & \text{if } 0 \leq i + j \leq n \\ x_{i+j-(n+1)}q & \text{if } n + 1 \leq i + j \leq 2n \end{cases}$$

The (small) quantum cohomology algebra is therefore

$$QH^*(\mathbf{CP}^n; \mathbf{C}) \cong \mathbf{C}[p, q, q^{-1}] / \langle qq^{-1} - 1, p^{n+1} - q \rangle$$

where we have written p for x_1 , and 1 for x_0 . The subalgebra

$$\widetilde{QH}^*(\mathbf{CP}^n; \mathbf{C}) \cong \mathbf{C}[p, q] / \langle p^{n+1} - q \rangle$$

is obtained by considering only nonnegative powers of q .

Notice that $QH^*(\mathbf{CP}^n; \mathbf{C})$, in contrast to $(H^*(\mathbf{CP}^n; \mathbf{C}), \circ_t)$, is an infinite-dimensional vector space. However, $QH^*(\mathbf{CP}^n; \mathbf{C})$ contains the same information as $(H^*(\mathbf{CP}^n; \mathbf{C}), \circ_t)$, because of the “periodicity isomorphisms”

$$\times q : QH^i(\mathbf{CP}^n; \mathbf{C}) \rightarrow QH^{i+2n+2}(\mathbf{CP}^n; \mathbf{C}).$$

To illustrate this, we list additive generators of $QH^i(\mathbf{CP}^n; \mathbf{C})$ for various degrees i in the table below:

$$\begin{array}{ccccccccccccccccc} \dots & -2n-2 & -2n & \dots & -2 & 0 & 2 & \dots & 2n & 2n+2 & 2n+4 & \dots & 4n+2 & \dots \\ \dots & x_0q^{-1} & x_1q^{-1} & \dots & x_nq^{-1} & x_0 & x_1 & \dots & x_n & x_0q & x_1q & \dots & x_nq & \dots \end{array}$$

Notice also that the “obvious” product structure on $H^*(\mathbf{CP}^n; \mathbf{C}) \otimes \Lambda$ results in the algebra $\mathbf{C}[p]/\langle p^{n+1} \rangle \otimes \Lambda$; this is quite different from the algebra $\mathbf{C}[p] \otimes \Lambda/\langle p^{n+1} - q \rangle$ obtained from the quantum product.

We can now explain the (deceptively trivial) indirect calculation of the quantum cohomology of \mathbf{CP}^n , which is the one usually given in expositions of the subject. The point is that, for dimensional reasons, “the only nontrivial quantum product is $x_1 \circ x_n = q$ ”. For example, since $|q| = 2n + 2$, there can be no term involving q in the quantum product $x_i \circ x_j$ when $i + j < n$, so we must have $x_i \circ x_j = x_{i+j}$. From this and the formula $x_1 \circ x_n = q$, all other quantum products $x_i \circ x_j$ may be deduced, e.g. $x_2 \circ x_n = x_1^2 \circ x_n = (x_1 \circ x_1) \circ x_n = x_1 \circ (x_1 \circ x_n) = x_1q$. To establish the formula $x_1 \circ x_n = q$, one must prove that $\langle X_1 | X_n | X_n \rangle_1 = 1$. This is easy, as $\langle X_1 | X_n | X_n \rangle_1$ is the number of linear maps $\mathbf{CP}^1 \rightarrow \mathbf{CP}^n$ which “hit” generic representatives of X_1 (a hyperplane), X_n (represented by a point not on the hyperplane), X_n (represented by another point not on the hyperplane) at three prescribed points of \mathbf{CP}^1 . But this argument is valid only if we assume various properties of the quantum product, such as its associativity and the fact that it is a deformation of the cup product. On the other hand, if we calculate all Gromov-Witten invariants $\langle X_i | X_j | X_k \rangle_s$ directly by linear algebra, then we will (in a very inefficient manner) establish these properties of the quantum product for the case of \mathbf{CP}^n .

We conclude with some comments on the technical conditions which are needed to justify the naive definition of $\langle A | B | C \rangle_D$. First, a very general definition of $\langle A | B | C \rangle_D$ (and hence of the quantum product) is possible under the assumption that the (connected simply connected Kähler—or even merely symplectic) manifold M is “positive” in some sense, for example that $\langle c_1 TM, D \rangle > 0$ for each homotopy class $D \in \pi_2(M)$ which contains a holomorphic map $\mathbf{CP}^1 \rightarrow M$. A Fano manifold — that is, a manifold M for which the cohomology class $c_1 TM$ can be represented by a Kähler 2-form — is automatically positive in this sense. It can be shown that the quantum product is commutative and associative, when M is positive. So far, however, there is no guarantee that the Gromov-Witten invariant $\langle A | B | C \rangle_D$ can be *computed* by the naive formula given earlier in this section. For this, one needs an additional assumption, for example that M is “convex” in the sense that $H^1(\mathbf{CP}^1, f^* TM) = 0$ for all holomorphic $f : \mathbf{CP}^1 \rightarrow M$. Convexity implies in particular that Hol_D is a manifold, and that it has the “expected” dimension $n + \langle c_1 TM, D \rangle$. Homogeneous Kähler manifolds are convex, for example.

The series defining the quantum product will in general contain infinitely many powers

of q (both positive and negative). However, if there exist $D_1, \dots, D_r \in \pi_2(M)$ such that all holomorphically representable classes D are of the form $\sum_{i=1}^r n_i D_i$ with $n_i \geq 0$, then it follows from the positivity condition that each such series contains only a finite number of terms $q^D = q_1^{s_1} \dots q_r^{s_r}$, and that $s_i \geq 0$ in each case. Any simply connected homogeneous Kähler manifold satisfies this condition, as well as positivity and convexity. However, problems arise as soon as we contemplate nonhomogeneous manifolds — even for such simple examples as the Hirzebruch surfaces (see Appendix 2).

§4 LANDAU-GINZBURG POTENTIALS

References: [Si-Ti], [Be1]-[Be3], [RRW]

There is a rather surprising “analytic” description of the cohomology algebra of the Grassmannian $Gr_k(\mathbf{C}^n)$ (and certain other Hermitian symmetric spaces), which was discovered recently by physicists ([LVW]). Before stating this, we shall review the two standard descriptions, which are well known to topologists. We shall generally follow the notation of [Si-Ti].

The “algebraic” (Borel) description of this cohomology algebra is

$$H^*(Gr_k(\mathbf{C}^n); \mathbf{Z}) \cong \frac{\mathbf{Z}[u_1, \dots, u_k, v_1, \dots, v_{n-k}]^{\Sigma_{k,n-k}}}{\mathbf{Z}[u_1, \dots, u_k, v_1, \dots, v_{n-k}]_+^{\Sigma_n}}$$

where $\mathbf{Z}[u_1, \dots, u_k, v_1, \dots, v_{n-k}]^{\Sigma_{k,n-k}}$ denotes the polynomials which are (separately) symmetric in u_1, \dots, u_k and in v_1, \dots, v_{n-k} , and $\mathbf{Z}[u_1, \dots, u_k, v_1, \dots, v_{n-k}]_+^{\Sigma_n}$ denotes the polynomials of positive degree which are symmetric in $u_1, \dots, u_k, v_1, \dots, v_{n-k}$. Each of $u_1, \dots, u_k, v_1, \dots, v_{n-k}$ represents a cohomology class of degree 2 here. This description is equivalent to the slightly more geometrical description

$$H^*(Gr_k(\mathbf{C}^n); \mathbf{Z}) \cong \mathbf{Z}[c_1, \dots, c_k, s_1, \dots, s_{n-k}] / \langle c(\mathcal{V})c(\mathbf{C}^n/\mathcal{V}) - 1 \rangle$$

where

$$\begin{aligned} c(\mathcal{V}) &= 1 + c_1 + c_2 + \dots + c_k \\ c(\mathbf{C}^n/\mathcal{V}) &= 1 + s_1 + s_2 + \dots + s_{n-k} \end{aligned}$$

are the total Chern classes of the tautologous bundle \mathcal{V} (of rank k) and the quotient bundle \mathbf{C}^n/\mathcal{V} (of rank $n-k$) on $Gr_k(\mathbf{C}^n)$. The previous description is obtained by writing $1 + c_1 + c_2 + \dots + c_k = (1 + u_1) \dots (1 + u_k)$ and $1 + s_1 + s_2 + \dots + s_{n-k} = (1 + v_1) \dots (1 + v_{n-k})$, in accordance with the Splitting Principle (see chapter 4 of [Bo-Tu]).

Taking the components in each positive degree of the identity $c(\mathcal{V})c(\mathbf{C}^n/\mathcal{V}) = 1$, we obtain n relations between c_1, \dots, c_k and s_1, \dots, s_{n-k} . To express these relations, it is convenient to consider them as the first n in an infinite series of formal relations

$$(1 + c_1 + c_2 + \dots)(1 + s_1 + s_2 + \dots) = 1$$

obtained from the product of two formal power series. We can use the first $n - k$ relations to express s_1, \dots, s_{n-k} in terms of c_1, \dots, c_k ; the next k relations can then be written in the form

$$s_{n-k+1} = f_{n-k+1}(c_1, \dots, c_k), \dots, s_n = f_n(c_1, \dots, c_k)$$

for some polynomials f_{n-k+1}, \dots, f_n . It follows that

$$H^*(Gr_k(\mathbf{C}^n); \mathbf{Z}) \cong \mathbf{Z}[c_1, \dots, c_k]/\langle f_{n-k+1}, \dots, f_n \rangle.$$

Alternatively, we can use the first k relations to express c_1, \dots, c_k in terms of s_1, \dots, s_{n-k} ; the next $n - k$ relations are then of the form

$$c_{k+1} = g_{k+1}(s_1, \dots, s_{n-k}), \dots, c_n = g_n(s_1, \dots, s_{n-k})$$

for some polynomials g_{k+1}, \dots, g_n . We obtain

$$H^*(Gr_k(\mathbf{C}^n); \mathbf{Z}) \cong \mathbf{Z}[s_1, \dots, s_{n-k}]/\langle g_{k+1}, \dots, g_n \rangle.$$

Example: $M = \mathbf{C}P^n$.

Here $k = 1$ and we have $f_{n+1}(c_1) = (-1)^{n+1}c_1^{n+1}$. This gives the familiar description

$$H^*(\mathbf{C}P^n; \mathbf{Z}) \cong \mathbf{Z}[c_1]/\langle c_1^{n+1} \rangle.$$

Alternatively, in terms of s_1, \dots, s_n , we have a rather more complicated description

$$H^*(\mathbf{C}P^n; \mathbf{Z}) \cong \mathbf{Z}[s_1, \dots, s_n]/\langle g_2, \dots, g_{n+1} \rangle.$$

The relations $g_2 = 0, \dots, g_{n+1} = 0$ are equivalent to (but not identical to) $s_{i+1} = s_1 s_i$, $1 \leq i \leq n$, which of course agrees with the previous description.

Example: $M = Gr_2(\mathbf{C}^4)$.

Here it makes no difference whether we use c_1, c_2 or s_1, s_2 . Let us choose c_1, c_2 . Then the relations are

$$s_1 = -c_1$$

$$s_2 = -s_1 c_1 - c_2 = c_1^2 - c_2$$

$$s_3 = -s_2 c_1 - s_1 c_2 - c_3 = -c_1^3 + 2c_1 c_2 - c_3$$

$$s_4 = -s_3 c_1 - s_2 c_2 - s_1 c_3 - c_4 = c_1^4 - 3c_1^2 c_2 + c_2^2 + c_1 c_3 - c_4$$

Setting $c_3 = c_4 = 0$, we obtain

$$f_3(c_1, c_2) = -c_1^3 + 2c_1c_2, \quad f_4(c_1, c_2) = c_1^4 - 3c_1^2c_2 + c_2^2.$$

The cohomology algebra is therefore

$$H^*(Gr_2(\mathbf{C}^4); \mathbf{Z}) \cong \mathbf{Z}[c_1, c_2]/\langle f_3, f_4 \rangle.$$

Notice that $f_4 + c_1f_3 = c_2^2 - c_1^2c_2$, so we can replace f_4 by this to obtain the slightly simpler description

$$H^*(Gr_2(\mathbf{C}^4); \mathbf{Z}) \cong \mathbf{Z}[c_1, c_2]/\langle -c_1^3 + 2c_1c_2, c_2^2 - c_1^2c_2 \rangle.$$

This could have been obtained immediately from the identity $(1 + c_1 + c_2 + \dots)(1 + s_1 + s_2 + \dots) = 1$ by setting $c_i = s_i = 0$ for $i > 2$. However, we prefer not to do this, as the polynomials f_i and g_j are more convenient for the general theory.

Now we turn to the “geometrical” (Schubert) description of the cohomology algebra of $Gr_k(\mathbf{C}^n)$. The rank of the abelian group $H^*(Gr_k(\mathbf{C}^n); \mathbf{Z})$ is $\binom{n}{k}$. Additive generators may be parametrized by Young diagrams of the form

or (less picturesquely) by k -tuples $(\lambda_1, \dots, \lambda_k) \in \mathbf{Z}^k$ such that $0 \leq \lambda_k \leq \lambda_{k-1} \leq \dots \leq \lambda_1 \leq n - k$. Observe that there are $\binom{n}{k}$ such diagrams, because they are in one to one correspondence with choices of k distinct elements $\lambda_k + 1, \lambda_{k-1} + 2, \dots, \lambda_1 + k$ from the set $\{1, 2, \dots, n\}$. The Schubert variety $X(\lambda)$ is defined to be the subvariety

$$X(\lambda) = \{V \in Gr_k(\mathbf{C}^n) \mid \dim V \cap \mathbf{C}^{n-k+i-\lambda_i} \geq i, 1 \leq i \leq k\}$$

of $Gr_k(\mathbf{C}^n)$. It is an irreducible subvariety of complex codimension $\sum_{i=1}^k \lambda_i$, and therefore it defines a homology class. The Poincaré dual cohomology class, of degree $2 \sum_{i=1}^k \lambda_i$, will be denoted by $x(\lambda)$.

Giambelli's formula expresses $x(\lambda)$ in terms of the multiplicative generators s_1, \dots, s_{n-k} of the Borel description:

$$x(\lambda) = \begin{vmatrix} s_{\lambda_1} & s_{\lambda_1+1} & s_{\lambda_1+2} & \cdots & s_{\lambda_1+k-1} \\ s_{\lambda_2-1} & s_{\lambda_2} & s_{\lambda_2+1} & \cdots & s_{\lambda_2+k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{\lambda_k-k+1} & s_{\lambda_k-k+2} & s_{\lambda_k-k+3} & \cdots & s_{\lambda_k} \end{vmatrix}$$

(where s_i is defined to be zero if $i \notin \{0, 1, \dots, n-k\}$, and $s_0 = 1$). These polynomials in s_1, \dots, s_{n-k} (parametrized by λ) are known as Schur functions. As a particular case of this formula we have

$$s_j = x(j, 0, \dots, 0), \quad 1 \leq j \leq n-k$$

and the Poincaré dual Schubert variety is

$$X(j, 0, \dots, 0) = \{V \in Gr_k(\mathbf{C}^n) \mid \dim V \cap \mathbf{C}^{n-k+1-j} \geq 1\}.$$

Such varieties are known classically as the “special” Schubert varieties. We also have

$$c_j = (-1)^j x(1, \dots, 1, 0, \dots, 0) \text{ (with } j \text{ entries equal to 1)}, \quad 1 \leq j \leq k$$

(see [Si-Ti]).

Intersection of Schubert varieties (after translation by generic elements of $GL_n \mathbf{C}$) corresponds to multiplication of the corresponding cohomology classes — this is the Schubert calculus. In addition to the Giambelli formula, there is an explicit combinatorial formula for the product $x(\lambda)x(\lambda')$, called the Littlewood-Richardson rule. For the special products $x(j, 0, \dots, 0)x(\lambda')$ this is a classical formula, called Pieri's formula. Giambelli's formula and Pieri's formula together determine all the products $x(\lambda)x(\lambda')$.

The analytic (Landau-Ginzburg) description of the cohomology algebra is a re-interpretation of the Borel description. It is

$$H^*(Gr_k(\mathbf{C}^n); \mathbf{Z}) \cong \mathbf{Z}[c_1, \dots, c_k]/\langle dP \rangle,$$

where P is a certain \mathbf{C} -valued polynomial function of c_1, \dots, c_k . In other words, it is asserted that the \mathbf{C}^k -valued function (f_{n-k+1}, \dots, f_n) has a “primitive”. Although this might seem superficial, it turns out that the existence of such “Landau-Ginzburg potentials” is fundamental in homological geometry.

Example: $M = \mathbf{C}P^n$.

A Landau-Ginzburg potential here is $P(c_1) = \pm c_1^{n+2}/(n+2)$.

Example: $M = Gr_2(\mathbf{C}^4)$.

A Landau-Ginzburg potential here is

$$P(c_1, c_2) = \frac{1}{5}(c_1^5 - 5c_1^3c_2 + 5c_1c_2^2),$$

as one verifies by differentiation:

$$\begin{aligned}\frac{\partial P}{\partial c_1} &= c_1^4 - 3c_1^2c_2 + c_2^2 = f_4 \\ \frac{\partial P}{\partial c_2} &= -c_1^3 + 2c_1c_2 = f_3.\end{aligned}$$

The function P first arose in physics, and from a mathematical point of view it is quite mysterious. However — with hindsight — it is easy to establish the existence of P , as follows. Consider the formal power series

$$\begin{aligned}C(t) &= 1 + c_1t + c_2t^2 + \dots \\ S(t) &= 1 + s_1t + s_2t^2 + \dots\end{aligned}$$

with $C(t)S(t) = 1$. Let

$$\log C(t) = W(t) = w_1t + w_2t^2 + \dots$$

We have

$$\frac{\partial}{\partial c_i} W(t) = C(t)^{-1} \frac{\partial}{\partial c_i} C(t) = S(t)t^i.$$

Equating the coefficients of t^{n+1} here, we obtain

$$\frac{\partial w_{n+1}}{\partial c_i} = s_{n+1-i}, \quad (1 \leq i \leq k).$$

Interpreting this in the Borel description of the cohomology algebra, we see that a Landau-Ginzburg potential is just

$$P = w_{n+1}.$$

We can express this more explicitly in terms of the original variables u_1, \dots, u_k . Recall that these were defined by $1 + c_1 + c_2 + \dots + c_k = (1 + u_1)\dots(1 + u_k)$. We have

$$\begin{aligned}w_1t + w_2t^2 + \dots &= \log(1 + c_1t + c_2t^2 + \dots + c_kt^k) \\ &= \sum_{i=1}^k \log(1 + u_i t) \\ &= \sum_{i=1}^k \left(-u_i t + \frac{1}{2}u_i^2 t^2 - \dots\right),\end{aligned}$$

hence

$$P = w_{n+1} = \frac{(-1)^{n+1}}{n+1} \sum_{i=1}^k u_i^{n+1}.$$

Now we turn to quantum cohomology. An elementary calculation (making use of the properties of the quantum product — see [Si-Ti]) gives:

$$\widetilde{QH}^*(Gr_k(\mathbf{C}^n); \mathbf{Z}) \cong \mathbf{Z}[c_1, \dots, c_k, q]/\langle f_{n-k+1}, \dots, f_{n-1}, f_n + (-1)^{n-k}q \rangle.$$

Hence — by inspection — we have

$$\widetilde{QH}^*(Gr_k(\mathbf{C}^n); \mathbf{Z}) \cong \mathbf{Z}[c_1, \dots, c_k, q]/\langle d\tilde{P} \rangle,$$

where

$$\tilde{P} = P + (-1)^{n-k}c_1q = \frac{(-1)^{n+1}}{n+1} \sum_{i=1}^k u_i^{n+1} + (-1)^{n-k} \sum_{i=1}^k u_i.$$

Thus, we have a Landau-Ginzburg potential for quantum cohomology too.

A nontrivial application of Landau-Ginzburg potentials is the existence of an analytic formula for quantum products (and ordinary products). This can be expressed either as an integral or a sum of residues. Moreover, the formula generalizes to a (conjectural) formula for higher genus Gromov-Witten invariants, the “Formula of Vafa and Intriligator”.

One version of the formula is as follows (see [Be1]). Let $T(c_1, \dots, c_k)$ be a polynomial whose total degree is equal to the dimension of $Gr_k(\mathbf{C}^n)$. This is an integer multiple of the generating class, and the integer is given up to sign by

$$\pm \sum_x \frac{T(x)}{h(x)}$$

where the sum is over the (finite) set of values x such that $d\tilde{P}(x) = 0$, and where

$$h = \det \left(\frac{\partial^2 \tilde{P}}{\partial c_i \partial c_j} \right).$$

§5 HOMOLOGICAL GEOMETRY

References: [Gi-Ki], [Au2]-[Au4]

We shall consider a sequence of spaces, denoted (i)-(iv) below. As we progress, we shall make the transition from the algebra of homology/cohomology theory to the geometry of manifolds/varieties. In this way we enter the world of “homological geometry”, a world (see section 1 of [Gi1]) where one deals with functions (and differential geometric objects) defined on homology and cohomology vector spaces.

(i) The manifold M .

We assume that M satisfies the technical conditions of §3. To proceed further, we shall need the cohomology R -modules $H^*(M; R)$ for $R = \mathbf{Z}$, $R = \mathbf{C}$, and $R = \mathbf{C}^* = \mathbf{C} - \{0\} \cong \mathbf{C}/2\pi\sqrt{-1}\mathbf{Z}$.

(ii) The manifold B .

Let us fix additive bases as follows:

$$H_2(M; \mathbf{Z}) = \bigoplus_{i=1}^r \mathbf{Z}A_i$$

$$H^2(M; \mathbf{Z}) = \bigoplus_{i=1}^r \mathbf{Z}b_i$$

A typical element of $H_2(M; \mathbf{C})$ will be denoted by $\sum_{i=1}^r p_i A_i$. Thus, we regard p_i as the i -th “coordinate function” on $H_2(M; \mathbf{C})$. Similarly, a typical element of $H^2(M; \mathbf{C})$ will be denoted by $\sum_{i=1}^r t_i b_i$.

We introduce the complex algebraic torus

$$B = H^2(M; \mathbf{C}/2\pi\sqrt{-1}\mathbf{Z}) = \bigoplus_{i=1}^r \mathbf{C}^*[b_i] \cong \mathbf{C}^* \times \cdots \times \mathbf{C}^*$$

where $[b_i]$ is the element of $H^2(M; \mathbf{C}/2\pi\sqrt{-1}\mathbf{Z}) \cong H^2(M; \mathbf{C})/H^2(M; 2\pi\sqrt{-1}\mathbf{Z})$ corresponding to b_i . A typical element of B will be denoted by (q_1, \dots, q_r) ; thus we regard q_i as the i -th coordinate function on B .

Unfortunately the notation p_i, t_i, q_i here breaks with our tradition of using lower-case letters for cohomology classes. In the case of p_i there is really no problem as it is a linear

functional on $H_2(M; \mathbf{C})$, and therefore can be identified with an element of $H^2(M; \mathbf{C})$. To make this more concrete, let us choose $b_i = a_i^c$, so that $\langle b_i, A_j \rangle = \langle a_i^c, A_j \rangle = \langle a_i^c a_j, M \rangle = \delta_{ij}$. Then $\langle b_i, \cdot \rangle$ is just the i -th coordinate function p_i . In other words, the cohomology class $b_i \in H^2(M; \mathbf{C})$ corresponds to the coordinate function $p_i \in H_2(M; \mathbf{C})^*$.

In the case of q_i , which previously denoted a “formal variable”, we can justify the current functional interpretation as follows. Recall (from §2) that the two versions of quantum cohomology are related by “putting $q^D = e^{\langle t, D \rangle}$ ”, and hence “ $q_i = e^{\langle t, D_i \rangle}$ ”. More precisely, in the first version (using \circ) we regard q as a formal variable, whereas in the second version (using \circ_t) we regard q_i as the function $t \mapsto e^{\langle t, D_i \rangle}$ on $H^2(M; \mathbf{C})$. Let us choose the basis A_1, \dots, A_r to be of the type denoted previously (at the ends of §2 and §3) by D_1, \dots, D_r . Then our current notation q_i denotes the function

$$q_i : H^2(M; \mathbf{C}) / H^2(M; 2\pi\sqrt{-1}\mathbf{Z}) \rightarrow \mathbf{C}^*, \quad [t] \mapsto e^{\langle t, A_i \rangle},$$

which is (induced by) our earlier realization of the formal variable q_i as a function.

In the case of the notation $\sum_{i=1}^r t_i b_i$, we shall just have to ask the reader to bear in mind that t_i is a coordinate function on $H^2(M; \mathbf{C})$, not a cohomology class. This has the advantage, at least, that we can write $t = \sum_{i=1}^r t_i b_i$ for a general cohomology class in $H^2(M; \mathbf{C})$. (In contrast, we must be careful to avoid writing $p = \sum_{i=1}^r p_i A_i$ or $p = (p_1, \dots, p_r)$, because (p_1, \dots, p_r) is a homology class, and p is a cohomology class. Similarly, we shall avoid writing $q = (q_1, \dots, q_r)$, because (q_1, \dots, q_r) is an element of B . The case $r = 1$ will need special care!)

These remarks on notation may seem tiresome, but we want to emphasize that we shall regard p_i and q_i primarily as *coordinate functions* on $H_2(M; \mathbf{C})$ and B (respectively), from now on. As is usual, we shall then write $\partial/\partial p_i$, $\partial/\partial q_i$ for the vector fields associated to these coordinate functions.

(iii) The manifold T^*B .

As B is a group, we have canonical isomorphisms

$$\begin{aligned} TB &\cong B \times H^2(M; \mathbf{C}) \\ T^*B &\cong B \times H^2(M; \mathbf{C})^*. \end{aligned}$$

Like all cotangent bundles, T^*B has a natural symplectic form $d\lambda$, where λ is the Liouville form. The symplectic manifold T^*B will be the focus of our attention in this section. We have

$$T^*B \cong B \times H_2(M; \mathbf{C})$$

if we use the above identification

$$H^2(M; \mathbf{C}) \cong H_2(M; \mathbf{C})^*, \quad b_i \mapsto p_i = \langle b_i, \cdot \rangle.$$

Then p_i and q_i may be regarded as \mathbf{C} -valued functions on the manifold T^*B . With this terminology, the 1-form λ is given explicitly by

$$\lambda = \sum_{i=1}^r p_i \wedge \frac{dq_i}{q_i}.$$

Next we introduce the group algebra

$$\begin{aligned} \Lambda &= \mathbf{C}[H_2(M; \mathbf{Z})] \\ &= \{\text{polynomials in } q_1, \dots, q_r, q_1^{-1}, \dots, q_r^{-1}\} \\ &= \{\text{regular functions on } B\} \end{aligned}$$

and the symmetric algebra

$$\begin{aligned} S &= S(H^2(M; \mathbf{C})) \\ &\cong S(H_2(M; \mathbf{C})^*) \\ &= \{\text{polynomials in } p_1, \dots, p_r\} \\ &= \{\text{regular functions on } H_2(M; \mathbf{C})\}. \end{aligned}$$

It follows that the algebra $S \otimes \Lambda$ may be identified with the algebra of regular functions on T^*B .

For the rest of this section we shall *assume* that “ $H^2(M; \mathbf{Z})$ generates $H^*(M; \mathbf{Z})$ ”, so that the natural homomorphism

$$S \rightarrow H^*(M; \mathbf{C})$$

is surjective. Hence $H^*(M; \mathbf{C})$ has the form

$$H^*(M; \mathbf{C}) \cong S/I \cong \mathbf{C}[p_1, \dots, p_r]/\langle R_1, R_2, \dots \rangle$$

where the ideal I (generated by R_1, R_2, \dots) is the kernel of the above homomorphism. The manifolds $\mathbf{C}P^n$ and $F_n = F_{1,2,\dots,n-1}(\mathbf{C}^n)$ satisfy this assumption, but $Gr_k(\mathbf{C}^n)$ (for $2 \leq k \leq n-2$) does not.

It follows that the natural homomorphism

$$S \otimes \Lambda \rightarrow QH^*(M; \mathbf{C})$$

is surjective as well, and hence (by Theorem 2.2 of [Si-Ti])

$$QH^*(M; \mathbf{C}) \cong S \otimes \Lambda / \mathcal{I} \cong \mathbf{C}[p_1, \dots, p_r, q_1, q_1^{-1}, \dots, q_r, q_r^{-1}] / \langle \mathcal{R}_1, \mathcal{R}_2, \dots \rangle$$

for some relations $\mathcal{R}_1, \mathcal{R}_2, \dots$ which are “quantum versions” of R_1, R_2, \dots .

(iv) The variety V_M .

Since $S \otimes \Lambda$ is the “coordinate ring” of T^*B , the quotient ring $S \otimes \Lambda / \mathcal{I}$ should be the coordinate ring of a subvariety V_M of T^*B . More informally, we shall just consider V_M to be the subvariety of T^*B defined by the equations $\mathcal{R}_i = 0$, i.e.

$$V_M = \{(q_1, \dots, q_r, p_1, \dots, p_r) \in T^*B \mid \mathcal{R}_1 = \mathcal{R}_2 = \dots = 0\}$$

It is shown in [Gi-Ki] and [Au4] that, under certain conditions, V_M is a Lagrangian subvariety of T^*B . This means that V_M is maximal isotropic with respect to the symplectic form $d\lambda$. We shall discuss this result in more detail in §7. For the moment, we consider two related properties:

$$(L1) \quad \mathcal{R}, \mathcal{S} \in \mathcal{I} \implies \{\mathcal{R}, \mathcal{S}\} \in \mathcal{I}$$

$$(L2) \quad \mathcal{R}, \mathcal{S} \in \mathcal{I} \implies \{\mathcal{R}, \mathcal{S}\} = 0$$

where $\{ , \}$ is the Poisson bracket associated to the symplectic form $d\lambda$.

To explain the relevance of these properties, we need to review some facts on symplectic geometry (see [We]). First, any (smooth) function $f : T^*B \rightarrow \mathbf{C}$ has an associated “Hamiltonian” vector field H_f , defined by $H_f = (d\lambda)^{-1} \circ df$. The Poisson bracket $\{f, g\}$ of two such functions is defined by $\{f, g\} = d\lambda(H_g, H_f)$ (this is equal to $dg(H_f)$ or $-df(H_g)$). In terms of the coordinates $p_1, \dots, p_r, q_1, \dots, q_r$ we have

$$\{f, g\} = - \sum_{i=1}^r q_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} \right)$$

We shall assume that (the smooth part of) V_M is a regular level set of the functions in \mathcal{I} , i.e. that

$$T_m V_M = \bigcap_{\mathcal{S} \in \mathcal{I}} \text{Ker } d\mathcal{S}_m$$

for all smooth points m .

Now, let us assume in addition that property (L1) holds. Then we have

$$d\mathcal{S}(H_{\mathcal{R}})|_{V_M} = \{\mathcal{R}, \mathcal{S}\}|_{V_M} = 0$$

for all $\mathcal{R}, \mathcal{S} \in \mathcal{I}$, so each vector field $H_{\mathcal{R}}$ is tangent to V_M . From the well known formula $[H_g, H_f] = H_{\{f, g\}}$, and property (L1) again, it follows that

$$[H_{\mathcal{R}}|_{V_M}, H_{\mathcal{S}}|_{V_M}] = [H_{\mathcal{R}}, H_{\mathcal{S}}]|_{V_M} = H_{\{\mathcal{S}, \mathcal{R}\}}|_{V_M} = 0$$

for all $\mathcal{R}, \mathcal{S} \in \mathcal{I}$. Hence, the vector fields $H_{\mathcal{R}}|_{V_M}$ define an integrable distribution — with “linear” leaves — on the smooth part of V_M . Any integral manifold V is isotropic with respect to $d\lambda$, since $d\lambda(H_{\mathcal{R}}, H_{\mathcal{S}})|_V = \{\mathcal{S}, \mathcal{R}\}|_V = 0$.

In general, V is strictly smaller than V_M ; in fact if \mathcal{I} is generated by relations $\mathcal{R}_1, \dots, \mathcal{R}_{r'}$, then we have $\dim V \leq r'$ and $\text{codim } V_M \leq r'$. However, in the special case where $V = V_M$ (and in particular $r = r' = \dim B$), then our discussion shows that property (L1) implies that V_M is isotropic and in fact maximal isotropic, i.e. Lagrangian.

As a temporary substitute for the general proof that V_M is Lagrangian, we shall verify property (L1) for several examples. In addition, we shall consider whether or not property (L2) is satisfied.

Example: $M = \mathbf{C}P^n$.

We have $V_M = \{(q, p) \in \mathbf{C}^* \times \mathbf{C} \mid p^{n+1} = q\}$. Thus V_M is a smooth submanifold of $\mathbf{C}^* \times \mathbf{C}$ (it is isomorphic to \mathbf{C}^*). Being one-dimensional, it is automatically Lagrangian. Property (L1), and indeed the stronger property (L2), is automatically satisfied in this example.

Example: $M = F_{1,2}(\mathbf{C}^3)$.

From Appendix 1, V_M is the subvariety of

$$T^*B = \{(q_1, q_2, x_1, x_2, x_3) \in (\mathbf{C}^*)^2 \times \mathbf{C}^3 \mid x_1 + x_2 + x_3 = 0\} \cong (\mathbf{C}^*)^2 \times \mathbf{C}^2$$

defined by the equations

$$\begin{aligned} \mathcal{R}_1 &= x_1x_2 + x_2x_3 + x_3x_1 + q_1 + q_2 = 0 \\ \mathcal{R}_2 &= x_1x_2x_3 + x_3q_1 + x_1q_2 = 0 \end{aligned}$$

We shall now make the following change of coordinates:

$$p_1 = x_1, \quad p_2 = x_1 + x_2$$

(this is dictated by our choice of isomorphism $T^*B \cong B \times H_2(M; \mathbf{C})$, i.e. we require $\langle p_i, A_j \rangle = \delta_{ij}$, where the q_i 's are defined relative to the basis consisting of the A_i 's). Then we obtain

$$\begin{aligned} \mathcal{R}_1 &= -p_1^2 - p_2^2 + p_1 p_2 + q_1 + q_2 \\ \mathcal{R}_2 &= -p_1 p_2^2 + p_1^2 p_2 - p_2 q_1 + p_1 q_2 \end{aligned}$$

Computing derivatives, we have

$$\begin{aligned} \frac{\partial \mathcal{R}_1}{\partial p_1} \frac{\partial \mathcal{R}_2}{\partial q_1} - \frac{\partial \mathcal{R}_2}{\partial p_1} \frac{\partial \mathcal{R}_1}{\partial q_1} &= (-2p_1 + p_2)(-p_2) - (-p_2^2 + 2p_1 p_2 + q_2) = -q_2 \\ \frac{\partial \mathcal{R}_1}{\partial p_2} \frac{\partial \mathcal{R}_2}{\partial q_2} - \frac{\partial \mathcal{R}_2}{\partial p_2} \frac{\partial \mathcal{R}_1}{\partial q_2} &= (-2p_2 + p_1)(p_1) - (-2p_1 p_2 + p_1^2 - q_1) = q_1, \end{aligned}$$

so

$$\{\mathcal{R}_1, \mathcal{R}_2\} = -q_1(-q_2) - q_2(q_1) = 0.$$

Thus, the stronger condition (L2) is satisfied in this case.

Example: $M = \Sigma_1 = \mathbf{P}(\mathcal{O}(0) \oplus \mathcal{O}(-1))$.

With the notation of Appendix 2, the quantum cohomology algebra of the Hirzebruch surface Σ_1 is

$$\widetilde{QH}^*(\Sigma_1; \mathbf{C}) \cong \frac{\mathbf{C}[x_1, x_4, q_1, q_2]}{\langle x_1^2 - x_2 q_2, x_4^2 - z - q_1 \rangle}$$

where $x_2 = x_4 - x_1$ and $z = x_1 x_4$. We introduce the new notation

$$p_1 = x_4, \quad p_2 = x_1$$

(again this is dictated by our choice of isomorphism $T^*B \cong B \times H_2(M; \mathbf{C})$), and obtain the relations

$$\begin{aligned} \mathcal{R}_1 &= p_2^2 - (p_1 - p_2)q_2 \\ \mathcal{R}_2 &= p_1^2 - p_1 p_2 - q_1. \end{aligned}$$

Computing derivatives, we have

$$\begin{aligned} \frac{\partial \mathcal{R}_1}{\partial p_1} \frac{\partial \mathcal{R}_2}{\partial q_1} - \frac{\partial \mathcal{R}_2}{\partial p_1} \frac{\partial \mathcal{R}_1}{\partial q_1} &= (-q_2)(-1) - (2p_1 - p_2)(0) = q_2 \\ \frac{\partial \mathcal{R}_1}{\partial p_2} \frac{\partial \mathcal{R}_2}{\partial q_2} - \frac{\partial \mathcal{R}_2}{\partial p_2} \frac{\partial \mathcal{R}_1}{\partial q_2} &= (2p_2 + q_2)(0) - (-p_1)(-p_1 + p_2) = -p_1^2 + p_1 p_2, \end{aligned}$$

so

$$\{\mathcal{R}_1, \mathcal{R}_2\} = -q_1(q_2) - q_2(-p_1^2 + p_1 p_2) = q_2 \mathcal{R}_2.$$

Thus, condition (L1) holds, but the stronger condition (L2) is *not* satisfied in this case.

As an indication of the importance of the Lagrangian subvariety V_M we shall discuss an observation of Givental and Kim ([Gi-Ki]) concerning the case $M = F_{1,2,\dots,n-1}(\mathbf{C}^n)$. Lagrangian manifolds appear naturally in the theory of completely integrable Hamiltonian systems, and it turns out for this M that V_M is such an example — the integrable system being the “one-dimensional Toda lattice”, or 1DTL for short.

For the case $M = F_{1,2}(\mathbf{C}^3)$, the observation will follow immediately from the calculation of $\widetilde{QH}^*(F_{1,2}(\mathbf{C}^3); \mathbf{C})$ in Appendix 1, once we have given the definition of the 1DTL. To end this section, therefore, we shall sketch the theory of the 1DTL for $n = 3$. A more detailed but very elementary discussion may be found in [Gu2].

The 1DTL is a system of first order o.d.e. in $2(n - 1)$ functions of $t \in \mathbf{R}$. For $n = 3$ this system is

$$\begin{aligned}\dot{a}_1 &= a_1(b_1 - b_2) \\ \dot{a}_2 &= a_2(b_2 - b_3)\end{aligned}$$

$$\begin{aligned}\dot{b}_1 &= -a_1 \\ \dot{b}_2 &= -a_2 + a_1 \\ \dot{b}_3 &= \quad a_2\end{aligned}$$

where $b_1 + b_2 + b_3 = 0$ and $a_1, a_2 > 0$. With $u_i = \log a_i$ this can be written in the form

$$\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} e^{u_1} \\ e^{u_2} \end{pmatrix}.$$

However, a different matrix formulation reveals the interesting geometrical structure of the system, namely the “Lax equation”

$$\dot{X} = [X, Y]$$

where

$$X = \begin{pmatrix} b_1 & a_1 & 0 \\ 1 & b_2 & a_2 \\ 0 & 1 & b_3 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we define

$$B = \{(t_1, t_2) \in \mathbf{R}^2 \mid t_1, t_2 > 0\}$$

we may identify T^*B as the “phase space”

$$T^*B \cong \left\{ \begin{pmatrix} s_1 & t_1 & 0 \\ 1 & s_2 & t_2 \\ 0 & 1 & s_3 \end{pmatrix} \text{ such that } (t_1, t_2) \in B, s_1, s_2, s_3 \in \mathbf{R}, s_1 + s_2 + s_3 = 0 \right\}$$

of the system. This is of course a (real) symplectic manifold. It can be shown that the o.d.e. $\dot{X} = [X, Y]$ describes the integral curves of a Hamiltonian vector field on T^*B .

Now, the form of the Lax equation implies that the solutions X are of the form $X(t) = A(t)V A(t)^{-1}$ for some function $A : \mathbf{R} \rightarrow GL_3 \mathbf{R}$. Without loss of generality we have $A(0) = I$ and so $X(0) = V$. In fact, it is possible to find A explicitly² and thus the solution of the original o.d.e., but we shall not need the explicit formula here. We just need the observation that *the function $\det(X(t) + \lambda I)$ is independent of t* . Indeed, we have $\det X(t) + \lambda I = \det V + \lambda I$ for any λ , hence the coefficients of the powers of λ are “conserved quantities” along each solution $X(t)$. These coefficients are as follows:

$$\begin{aligned} \lambda^3 &: 1 \\ \lambda^2 &: b_1 + b_2 + b_3 (= 0) \\ \lambda^1 &: b_1 b_2 + b_2 b_3 + b_3 b_1 - a_1 - a_2 (= g(a, b), \text{ say}) \\ \lambda^0 &: b_1 b_2 b_3 - b_3 a_1 - b_1 a_2 (= h(a, b), \text{ say}) \end{aligned}$$

In classical language, the two nontrivial conserved quantities g and h are “first integrals” of the system, and they lead to (another) method of finding the solution: substitute $a_1 = -\dot{b}_1$, $a_2 = \dot{b}_3$ into the equations

$$\begin{aligned} g(a, b) &= C_1 \\ h(a, b) &= C_2 \end{aligned}$$

(for constants C_1, C_2) and then try to solve the reduced system of first order o.d.e for b_1, b_2, b_3 .

In modern language, the equations $g(a, b) = C_1$ and $h(a, b) = C_2$ define a “Lagrangian leaf” of the foliation of T^*B given by the corresponding Hamiltonian vector fields H_g, H_h . Each solution curve X lies entirely within such a leaf.

²Namely: $A(t)$ is the matrix obtained by applying the Gram-Schmidt orthogonalization process to the columns of the matrix $\exp tV$. From this one obtains expressions for $a_i(t)$ and $b_i(t)$ as rational functions of exponential functions.

Comparison with our earlier discussion of $M = F_{1,2}(\mathbf{C}^3)$ reveals that this Lagrangian leaf — for $C_1 = C_2 = 0$, and in the complex variable case — is precisely the manifold V_M .

A complete proof of this surprising coincidence in the case of $n \times n$ matrices — which was sketched only rather briefly in [Gi-Ki] — can be found in [Ci] and [Ki]. The main point is to prove that the conserved quantities of the 1DTL are relations in the quantum cohomology ring of $F_{1,2,\dots,n-1}(\mathbf{C}^n)$. The proofs given in these references all use indirect methods (with a view to future generalizations). An elementary direct proof can be found in [Gu-Ot], together with the sketch of a generalization to the case of the (infinite dimensional) “periodic flag manifold” and the periodic 1DTL.

§6 A SYSTEM OF DIFFERENTIAL OPERATORS

References: [Gi-Ki], [Au3]

We assume as in §5 that $H^2(M; \mathbf{Z}) = \bigoplus_{i=1}^r \mathbf{Z} b_i$ generates $H^*(M; \mathbf{Z})$. Writing $t = \sum_{i=1}^r t_i b_i$, we have in this situation a generating function for the (finitely many) products $b_1^{j_1} \dots b_r^{j_r}$ in $H^*(M; \mathbf{Z})$:

$$v(t) = e^t = \sum_{l \geq 0} \frac{1}{l!} t^l = \sum_{j, l \text{ s.t. } \sum j_i = l} \frac{t_1^{j_1} \dots t_r^{j_r}}{j_1! \dots j_r!} b_1^{j_1} \dots b_r^{j_r}.$$

Evidently

$$\left. \frac{\partial^{j_1 + \dots + j_r}}{\partial t_1^{j_1} \dots \partial t_r^{j_r}} v(t) \right|_0 = b_1^{j_1} \dots b_r^{j_r},$$

i.e. the products are given explicitly by the derivatives of the generating function v . There is also a “scalar” generating function

$$V(t) = \langle v(t), M \rangle,$$

which carries the same information as v .

Example: $M = \mathbf{C}P^n$.

We have $v(t) = 1 + tb + \frac{1}{2!}t^2b^2 + \dots + \frac{1}{n!}t^n b^n$, where $b = x_1$. The scalar generating function is just $V(t) = \frac{1}{n!}t^n$.

Example: $M = F_{1,2}(\mathbf{C}^3)$.

Using the notation of Appendix 1, we have: $v(t) = 1 + t_1a + t_2b + \frac{1}{2!}(t_1^2a^2 + 2t_1t_2ab + t_2^2b^2) + \frac{1}{3!}(3t_1^2t_2a^2b + 3t_1t_2^2ab^2)$. The scalar generating function is $V(t) = \frac{1}{3!}(3t_1^2t_2 + 3t_1t_2^2)$.

For quantum products we can make a similar definition. Namely, we define

$$v(t, q) = \sum_{l \geq 0} \frac{1}{l!} t \circ \cdots \circ t$$

where the indicated product $t \circ \cdots \circ t$ is the quantum product of l copies of t . We also have

$$V(t, q) = \langle v(t, q), M \rangle$$

We shall not discuss the convergence of this formal series. Nevertheless, as in the case of ordinary cohomology, we have (formally, at least), the following equation for the quantum products:

$$\left. \frac{\partial^{j_1+\cdots+j_r}}{\partial t_1^{j_1} \cdots \partial t_r^{j_r}} v(t) \right|_0 = (b_1 \circ \cdots \circ b_1) \circ \cdots \circ (b_r \circ \cdots \circ b_r),$$

where b_1, \dots, b_r appear (respectively) j_1, \dots, j_r times on the right hand side.

Example: $M = \mathbf{C}P^n$.

We have $v(t, q) = \sum_{l \geq 0} \frac{1}{l!} t^l b \circ \cdots \circ b$ (with $b = x_1$), and $V(t, q) = \sum_{s \geq 0} \frac{t^{(n+1)s+n}}{((n+1)s+n)!} q^s$.

Now we introduce the differential operators referred to in the title of this section.

Definition. For any $R \in S \otimes \Lambda$, we define a differential operator R^* by $R^* = R\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r}, q_1, q_1^{-1}, \dots, q_r, q_r^{-1}\right)$.

In other words, to produce R^* from R , we identify the coordinate function p_i with the cohomology class b_i , and then regard this as the vector field $\partial/\partial t_i$ on $H^2(M; \mathbf{C})$. The following observation of [Gi-Ki] is also explained in [Au3] (Theorem 3.3.1).

Theorem. $R^*V(t, q) = 0 \iff R \in \mathcal{I}$.

Proof. We have

$$\begin{aligned}
R^*V = 0 &\iff \left. \frac{\partial^{j_1 + \dots + j_r}}{\partial t_1^{j_1} \dots \partial t_r^{j_r}} R^*V \right|_0 = 0 \text{ for all } j \\
&\iff (p_1^{j_1} \dots p_r^{j_r} R)^* V|_0 = 0 \text{ for all } j \\
&\iff \langle (p_1^{j_1} \dots p_r^{j_r} R)^* v, M \rangle|_0 = 0 \text{ for all } j \\
&\iff \langle b_1^{j_1} \dots b_r^{j_r} R, M \rangle = 0 \text{ for all } j
\end{aligned}$$

where $b_1^{j_1} \dots b_r^{j_r} R$ is interpreted as an expression in quantum cohomology, i.e. all products are quantum products. By the nondegeneracy of the intersection form, the last condition is equivalent to the vanishing of R (considered as a quantum polynomial), i.e. to the statement $R \in \mathcal{I}$.

This result is somewhat tautological, since the generating function V clearly contains by definition all the quantum products. However, it gives an interesting new point of view on the quantum cohomology algebra: it is the algebra of differential operators which annihilate the generating function V .

§7 THE ROLE OF THE FLAT CONNECTION

References: [Gi-Ki], [Gi1]-[Gi6]

In sections §4-§6 we discussed three “analytic” aspects of quantum cohomology. The key to understanding these is the family of flat connections $\nabla^\lambda = d + \lambda\omega$ of §1, with $\omega_t(X)(Y) = X \circ_t Y$.

Flatness of ∇^λ via a generating function. In §2 we stated that ∇^λ is a flat connection, for each λ . Flatness of ∇^λ (for all λ) is equivalent to the conditions

$$d\omega = 0, \quad \omega \wedge \omega = 0$$

(see §1). The second condition follows from the commutativity and associativity of \circ_t . To establish the first condition, it suffices to find a function $K : H^2(M; \mathbf{C}) \rightarrow \text{End}(W)$ such that $\omega = dK$.

Before proceeding with this, we remark that there is an obvious generalization of the triple products $\langle A|B|C \rangle_D$, namely

$$\langle A_1 | \dots | A_i \rangle_D = |\text{Hol}_D^{\tilde{A}_1, p_1} \cap \dots \cap \text{Hol}_D^{\tilde{A}_i, p_i}|$$

for any $i \geq 3$. It can be shown (with the appropriate technical assumptions, cf. §3) that

$$a_1 \circ \cdots \circ a_i = \sum_{D \in H_2(M; \mathbf{Z})} (a_1 \circ \cdots \circ a_i)_D q^D$$

where

$$\langle (a_1 \circ \cdots \circ a_i)_D, B \rangle = \langle A_1 | \dots | A_i | B \rangle_D$$

for all homology classes B ; hence these i -fold products are determined³ by the quantum products, and therefore by the 3-fold (triple) products, for $i \geq 3$. Using this notation, the generating function $V(t, q)$ of §6 is (ignoring the “quadratic part”)

$$V(t, q) = \sum_{l \geq 3; D} \frac{1}{l!} \langle T | \dots | T \rangle_D q^D$$

where T (the Poincaré dual of $t = \sum_{i=1}^r t_i b_i$) appears l times in the indicated term.

A slight modification gives a definition of 2-fold products, namely

$$\langle A_1 | A_2 \rangle_D = |(\text{Hol}_D^{\tilde{A}_1, p_1} \cap \text{Hol}_D^{\tilde{A}_2, p_2}) / \mathbf{C}^*|$$

where \mathbf{C}^* is identified with the subgroup of linear fractional transformations (of \mathbf{CP}^1) which fix p_1 and p_2 . These products are also related to the triple products, because of the “divisor property”

$$\langle A_1 | A_2 | X \rangle_D = \langle A_1 | A_2 \rangle_D \langle x, D \rangle$$

where x is any element of $H^2(M; \mathbf{Z})$ and $D \neq 0$. (Roughly speaking, this formula follows from the fact that a nonconstant holomorphic map $f : \mathbf{CP}^1 \rightarrow M$ in the homotopy class D must hit a subvariety X of complex codimension 1 in $\langle x, D \rangle$ points — this is a topological necessity, beyond the influence of A_1, A_2 .)

Let us return now to the function K . We claim that the following explicit formula defines a function with the desired property $\omega = dK$:

$$\langle K(t)(a), C \rangle = \langle A | C | T \rangle_0 + \sum_{D \neq 0} \langle A | C \rangle_D e^{\langle t, D \rangle}$$

where $t \in H^2(M; \mathbf{Z})$ and $a \in H^*(M; \mathbf{Z})$ are fixed, and where C is a general element of $H_*(M; \mathbf{Z})$. The function K may be extended by complex linearity to the case of $t \in H^2(M; \mathbf{C})$ and $a \in H^*(M; \mathbf{C})$.

³There is another kind of i -fold product in which the basepoints p_1, \dots, p_i are allowed to vary (and which are of more interest, from the point of view of enumerative geometry). These i -fold products are not in general determined by the triple products, and the situation is more complicated. For $i = 3$, both definitions coincide.

Having produced a candidate for K , the proof of the claim will be an elementary verification. We choose $x \in H^2(M; \mathbf{C})$ and regard it as a vector field on $H^2(M; \mathbf{C})$. Using the differentiation formula $xe^{\langle t, D \rangle} = \langle x, D \rangle e^{\langle t, D \rangle}$, we obtain:

$$\begin{aligned} x \cdot \langle K(t)(a), C \rangle &= x \cdot \langle act, M \rangle + \sum_{D \neq 0} \langle A|C\rangle_D x \cdot e^{\langle t, D \rangle} \\ &= \langle acx, M \rangle + \sum_{D \neq 0} \langle A|C\rangle_D \langle x, D \rangle e^{\langle t, D \rangle} \\ &= \langle A|C|X\rangle_0 + \sum_{D \neq 0} \langle A|C|X\rangle_D e^{\langle t, D \rangle} \\ &= \langle x \circ_t a, C \rangle. \end{aligned}$$

Hence $dK(t)(a)(x) = xK(t)(a) = x \circ_t a = \omega_t(a)(x)$, i.e. $dK = \omega$, as required.

Generating functions as solutions of integrable systems. The flatness of the connection $\nabla^\lambda = d + \lambda\omega$ (on the simply connected manifold W) implies that it is gauge-equivalent via some gauge transformation H to the trivial connection d . In other words, the equation $d\omega + \lambda\omega \wedge \omega = 0$ implies that $\lambda\omega = H^{-1}dH$ for some map $H : W \rightarrow Gl(W)$. (This is a standard fact from differential geometry. Note that H will depend on λ here.) It can be shown that “generating function” H characterizes the quantum product in a similar way to the generating function V of §6. This fact was used (for example) in [Ki] to establish a relation between the quantum cohomology of the flag manifold G/B and the Toda lattice associated to a Lie group G , generalizing the case $G = U_n$ which was discussed in §5. A brief explanation of this argument can be found in [Gi6].

Lagrangian subvarieties. Another application of the flat connection ∇^λ is the proof that the variety V_M of §5 is Lagrangian. We shall just sketch the argument (of [Gi-Ki], [Au4]) briefly.

The first ingredient is the subvariety Σ_M of T^*B defined by the “characteristic polynomials of the linear transformations $\omega_t(x) \in \text{End}(W)$, for $x \in H^2(M; \mathbf{C})$ ”. That is,

$$\Sigma_M = \{([t], \lambda) \in B \times H^2(M; \mathbf{C})^* \mid \det \omega_t(x) - \lambda(x)I = 0 \text{ for all } x \in H^2(M; \mathbf{C})\}.$$

Let us assume that the linear transformation $\omega_t(x)$ has $s+1 (= \dim W)$ distinct eigenvalues for at least one (and hence almost all) values of t . Then it can be shown that Σ_M and V_M coincide. The variety V_M therefore has $s+1$ branches, the i -th branch being given by the equations

$$p_1 = \lambda_i(b_1), \dots, p_r = \lambda_i(b_r)$$

where λ_i is the i -th eigenvalue. (Note that the entries of the matrix of $\omega_t(b_i)$ with respect to the basis b_0, \dots, b_s , and hence the eigenvalues $\lambda_i(b_j)$, are functions of q_1, \dots, q_r .)

Next, from the fact that $d\omega = 0$, it can be deduced that $d\lambda_i = 0$. The eigenvalue λ_i can be regarded as a 1-form on B , and the i -th branch of Σ_M is simply the image of this 1-form. But it is well known (see [We]) that the image of a closed 1-form is Lagrangian, so this completes the proof.

Landau-Ginzburg potentials. So far we have not mentioned the Landau-Ginzburg potentials of §4. It turns out that these are related to the mirror symmetry phenomenon. As explained in [Au2], mirror symmetry can be viewed as a correspondence between “Landau-Ginzburg models” (which produce Frobenius manifolds from singularity theory) and “field theory models” (with Frobenius manifolds given by quantum cohomology). We have seen that the quantum cohomology of the Grassmannian is related to the singularities of the Landau-Ginzburg potential \tilde{P} . A mirror principle for the flag manifold F_n is proposed in [Gi6].

APPENDIX 1: $QH^*(F_{1,2}(\mathbf{C}^3))$

We shall calculate — in obsessive detail — the quantum cohomology of the flag manifold

$$F_3 = F_{1,2}(\mathbf{C}^3) = \{(L, V) \in Gr_1(\mathbf{C}^3) \times Gr_2(\mathbf{C}^3) \mid L \subseteq V\}.$$

This is a complex manifold of real dimension 6 (so $n = 3$ in the terminology of §2 and §3).

The following “Borel description” of $H^*(F_3; \mathbf{Z})$ is well known:

$$H^*(F_3; \mathbf{Z}) \cong \mathbf{Z}[x_1, x_2, x_3]/\langle \sigma_1, \sigma_2, \sigma_3 \rangle$$

where $\sigma_1, \sigma_2, \sigma_3$ are the elementary symmetric functions of x_1, x_2, x_3 . Geometrically, $x_i = -c_1 \mathcal{L}_i$, where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are the complex line bundles on F_3 whose fibres over (L, V) are L , $L^\perp \cap V$, V^\perp respectively.

We may choose additive generators as follows:

$$\begin{array}{cccc} H^0(F_3; \mathbf{Z}) & H^2(F_3; \mathbf{Z}) & H^4(F_3; \mathbf{Z}) & H^6(F_3; \mathbf{Z}) \\ 1 & x_1 & x_1^2 & x_1^2 x_2 \\ & x_2 & x_1 x_2 & \end{array}$$

The remaining cup products are determined by the relations $x_1^2 + x_2^2 + x_1 x_2 = 0$, $x_1 x_2^2 + x_1^2 x_2 = 0$ (which imply $x_1^3 = 0, x_2^3 = 0$).

This description ignores the complex manifold structure of F_3 . Since we shall be considering holomorphic maps, it is more appropriate to use the “Schubert description” of $H^*(F_3; \mathbf{Z})$, which amounts to replacing x_1, x_2 by $a = x_1, b = x_1 + x_2$. Geometrically, $a = c_1 \mathcal{L}^*$ and $b = c_1 \mathcal{V}^*$, where \mathcal{L}, \mathcal{V} are the holomorphic “tautological” bundles whose fibres over (L, V) are L, V respectively. We then choose the following additive generators:

$$\begin{array}{cccc} H^0(F_3; \mathbf{Z}) & H^2(F_3; \mathbf{Z}) & H^4(F_3; \mathbf{Z}) & H^6(F_3; \mathbf{Z}) \\ 1 & a & a^2 & a^2 b = ab^2 \\ & b & b^2 & \end{array}$$

This time the remaining cup products are determined by $ab = a^2 + b^2$ (which implies $a^3 = 0, b^3 = 0$). From the definition of a, b we have $a^2 = x_1^2$, $b^2 = x_1x_2$, and $a^2b = x_1^2x_2$, so the only difference between the last table and the previous table is that x_2 has been replaced by b . From the table we have $a^c = b^2$, $b^c = a^2$, $(a^2)^c = b$, $(b^2)^c = a$.

With respect to a fixed reference flag $E_1 \subseteq E_2 \subseteq \mathbf{C}^3$ there are six Schubert varieties. Excluding the trivial cases “ F_3 ” and “a point”, we list these below, together with their Poincaré dual cohomology classes.

- (1) $\{L \subseteq E_2\} = \text{PD}(a) \in H_4(F_3; \mathbf{Z})$
- (2) $\{E_1 \subseteq V\} = \text{PD}(b) \in H_4(F_3; \mathbf{Z})$
- (3) $\{L = E_1\} = \text{PD}(a^2) \in H_2(F_3; \mathbf{Z})$
- (4) $\{V = E_2\} = \text{PD}(b^2) \in H_2(F_3; \mathbf{Z}).$

Of course, $\{L \subseteq E_2\}$ is an abbreviation for $\{(L, V) \in F_3 \mid L \subseteq E_2\}$, etc.

We are now ready to consider holomorphic maps $f : \mathbf{CP}^1 \rightarrow F_3$. We begin by choosing the following basis for $H_2(F_3; \mathbf{Z})$:

$$H_2(F_3; \mathbf{Z}) = \mathbf{Z}A^c \oplus \mathbf{Z}B^c \quad (\text{where } A^c = \text{PD}(a^c) = \text{PD}(b^2), B^c = \text{PD}(b^c) = \text{PD}(a^2)).$$

If $[f] = d_1A^c + d_2B^c$ for some $d_1, d_2 \in \mathbf{Z}$, then it follows⁴ that

$$d_1 = f^*a, \quad d_2 = f^*b$$

(with respect to a fixed choice of generator of $H^2(\mathbf{CP}^1; \mathbf{Z})$). It can be shown that the space $\text{Hol}_{d_1, d_2}^{F_3, p}$ is nonempty if and only if either (a) $d_2 \geq d_1 \geq 0$ or (b) $d_2 = 0, d_1 \geq 0$.

With respect to the above choice of basis we can write $q^{(d_1, d_2)} = q_1^{d_1}q_2^{d_2}$, where $q_1 = q^{A^c}$ and $q_2 = q^{B^c}$. We have $|q_1| = |q_2| = 4$. To calculate the quantum product \circ , we begin by investigating the numerical condition for $\langle X|Y|Z \rangle_D \neq 0$. It is known that $\langle c_1TF_3, D \rangle = 2d_1 + 2d_2$, so the numerical condition is

$$|x| + |y| + |z| = 6 + 4d_1 + 4d_2.$$

Since $\text{Hol}_{d_1, d_2}^{F_3, p}$ is empty when either of d_1 or d_2 is negative, we have

$$x \circ y = \sum_{d_1, d_2 \geq 0} (x \circ y)_{d_1, d_2} q_1^{d_1} q_2^{d_2}.$$

Since the degree $|(x \circ y)_{d_1, d_2}|$ is given by $|x| + |y| - 4d_1 - 4d_2$, and this must be 0, 2, 4, or 6, the relevant values of (d_1, d_2) are severely restricted. We shall calculate all possible quantum products of the additive basis elements a, b, a^2, b^2, a^2b .

⁴For example, $f^*a = \langle a, d_1A^c + d_2B^c \rangle = d_1\langle a, A^c \rangle + d_2\langle a, B^c \rangle = d_1$.

Proposition 1. $a \circ a = a^2 + q_1$.

Proof. We have $a \circ a = (a \circ a)_{0,0} + (a \circ a)_{1,0}q_1 + (a \circ a)_{0,1}q_2$. Now, $(a \circ a)_{0,0}$ is necessarily a^2 , so it remains to calculate the degree 0 cohomology classes $\lambda = (a \circ a)_{1,0}$ and $\mu = (a \circ a)_{0,1}$.

By definition, $\lambda = \langle A|A|Z \rangle_{1,0}$, where Z denotes the generator of $H_0(F_3; \mathbf{Z})$. We must look for three suitable representatives of the homology classes A, A, Z . We try

- (a) $\{L \subseteq E'_2\}$ (representing A)
- (b) $\{L \subseteq E''_2\}$ (also representing A)
- (c) (E_1, E_2) (a single point of F_3)

where E_1, E_2, E'_2, E''_2 are to be chosen (if possible) so that there are only finitely many holomorphic maps of degree $(1, 0)$ touching (a), (b) and (c) at three distinct points.

Now, any holomorphic map of degree $(1, 0)$ is of the form $\mathbf{P}(H) \rightarrow F_3$, $L \mapsto (L, H)$, where H is a fixed two-dimensional subspace of \mathbf{C}^3 . Therefore, a precise formulation of the problem is as follows: we must choose E_1, E_2, E'_2, E''_2 so that there are only finitely many configurations (L', L'', L''', H) — where L', L'', L''' are distinct lines in H — such that

- (a) $L' \subseteq E'_2$
- (b) $L'' \subseteq E''_2$
- (c) $L''' = E_1, H = E_2$

Let us choose the two-dimensional subspaces E_2, E'_2, E''_2 in general position, i.e. such that the intersection of any two of them is a line, and the intersection of all three is the origin. Let us choose E_1 to be any line in E_2 . Then there is a *single* configuration (L', L'', L''', H) satisfying (a), (b) and (c), namely

$$L' = H \cap E'_2, \quad L'' = H \cap E''_2, \quad L''' = E_1, \quad H = E_2.$$

We conclude that $\lambda = 1$.

Next we calculate $\mu = \langle A|A|Z \rangle_{0,1}$ by a similar method. The new feature is that holomorphic maps of degree $(0, 1)$ are of the form $\mathbf{P}(\mathbf{C}^3/K) \rightarrow F_3$, $V/K \mapsto (K, V)$, where K is a fixed line in \mathbf{C}^3 . So we must choose E_1, E_2, E'_2, E''_2 so that there are only finitely many configurations (K, V', V'', V''') — where V', V'', V''' are distinct two-dimensional subspaces containing the line K — such that

- (a) $K \subseteq E'_2$
- (b) $K \subseteq E''_2$
- (c) $K = E_1, V''' = E_2$.

By choosing E_2, E'_2, E''_2 in general position, and $E_1 \neq E'_2 \cap E''_2$, we see that there are *no* such configurations. We conclude that $\mu = 0$.

Proposition 2. $b \circ b = b^2 + q_2$.

Proof. Similar to the proof of Proposition 1.

Proposition 3. $a \circ b = ab$.

Proof. We have to show that $(a \circ b)_{1,0} = (a \circ b)_{0,1} = 0$. First we have $(a \circ b)_{1,0} = \langle A|B|Z \rangle_{1,0}$. As representatives of A, B, Z we try

- (a) $\{L \subseteq E_2\}$ (representing A)
- (b) $\{E'_1 \subseteq V\}$ (representing B)
- (c) (E''_1, E''_2) (a single point of F_3)

We must choose E'_1, E''_1, E_2, E''_2 so that there are only finitely many configurations (L', L'', L''', H) — where H is two-dimensional and L', L'', L''' are distinct lines in H — such that

- (a) $L' \subseteq E_2$
- (b) $E'_1 \subseteq H$
- (c) $L''' = E''_1, H = E''_2$

Let us choose E''_1, E_2, E''_2 arbitrarily, and E'_1 such that $E'_1 \not\subseteq E''_2$. Since (b) and (c) imply that $E'_1 \subseteq E''_2$, there are no such configurations. Hence $(a \circ b)_{1,0} = 0$. A similar calculation shows that $(a \circ b)_{0,1} = 0$.

Proposition 4. $a \circ b^2 = ab^2$.

Proof. We have $a \circ b^2 = ab^2 + (a \circ b^2)_{1,0}q_1 + (a \circ b^2)_{0,1}q_2$.

Let $(a \circ b^2)_{1,0} = \lambda a + \mu b$. We have $\lambda = \langle A|B^2|B^2 \rangle_{1,0}$. As representatives of the homology classes A, B^2, B^2 , we try:

- (a) $\{L \subseteq E_2\}$
- (b) $\{V = E'_2\}$
- (c) $\{V = E''_2\}$

We will choose E_2, E'_2, E''_2 so that there are only finitely many configurations (L', L'', L''', H) — where L', L'', L''' are distinct lines in H — such that

$$(a) \quad L' \subseteq E_2$$

$$(b) \quad H = E'_2$$

$$(c) \quad H = E''_2$$

Thus $E'_2 = E''_2$, but this is impossible if we choose E_2, E'_2, E''_2 in general position. Hence $\lambda = 0$.

Next, we have $\mu = \langle A|B^2|A^2 \rangle_{1,0}$. As representatives of the homology classes A, B^2, A^2 , we try:

$$(a) \quad \{L \subseteq E_2\}$$

$$(b) \quad \{V = E'_2\}$$

$$(c) \quad \{L = E''_1\}$$

We will choose E_2, E'_2, E''_1 so that there are only finitely many configurations (L', L'', L''', H) — where L', L'', L''' are distinct lines in H — such that

$$(a) \quad L' \subseteq E_2$$

$$(b) \quad H = E'_2$$

$$(c) \quad L''' = E''_1$$

Thus, $E''_1 \subseteq E'_2$, but we may choose E_2, E'_2, E''_2 so that this is false. Hence $\mu = 0$.

Now we turn to $(a \circ b^2)_{0,1} = \lambda a + \mu b$. We have $\lambda = \langle A|B^2|B^2 \rangle_{0,1}$. As representatives of the homology classes A, B^2, B^2 , we try:

$$(a) \quad \{L \subseteq E_2\}$$

$$(b) \quad \{V = E'_2\}$$

$$(c) \quad \{V = E''_2\}$$

We will choose E_2, E'_2, E''_2 so that there are only finitely many configurations (K, V', V'', V''') — where V', V'', V''' are distinct two-dimensional subspaces containing the line K — such that

$$(a) \quad K \subseteq E_2$$

$$(b) \quad V'' = E'_2$$

$$(c) \quad V''' = E_2''$$

Thus $K \subseteq E_2, E_2', E_2''$, but this is impossible if we choose E_2, E_2', E_2'' in general position. Hence $\lambda = 0$.

Next, we have $\mu = \langle A|B^2|A^2 \rangle_{0,1}$. As representatives of the homology classes A, B^2, A^2 , we try:

$$(a) \quad \{L \subseteq E_2\}$$

$$(b) \quad \{V = E_2'\}$$

$$(c) \quad \{L = E_1''\}$$

We will choose E_2, E_2', E_1'' so that there are only finitely many configurations (K, V', V'', V''') — where V', V'', V''' are distinct two-dimensional subspaces containing the line K — such that

$$(a) \quad K \subseteq E_2$$

$$(b) \quad V'' = E_2'$$

$$(c) \quad K = E_1''$$

Thus, $E_1'' \subseteq E_2$, but we may choose E_2, E_2', E_2'' so that this is false. Hence $\mu = 0$.

Proposition 5. $a^2 \circ b = a^2 b$.

Proof. Similar to the proof of Proposition 4.

Proposition 6. $a \circ a^2 = b q_1$.

Proof. We have $a \circ a^2 = (a \circ a^2)_{1,0} q_1 + (a \circ a^2)_{0,1} q_2$.

Let $(a \circ a^2)_{1,0} = \lambda a + \mu b$. We have $\lambda = \langle A|A^2|B^2 \rangle_{1,0}$, which is zero by the proof of Proposition 4.

Next, we have $\mu = \langle A|A^2|A^2 \rangle_{1,0}$. As representatives of the homology classes A, A^2, A^2 , we try:

$$(a) \quad \{L \subseteq E_2\}$$

$$(b) \quad \{L = E_1'\}$$

$$(c) \quad \{L = E_1''\}$$

We will choose E_2, E_1', E_1'' so that there are only finitely many configurations (L', L'', L''', H) — where L', L'', L''' are distinct lines in H — such that

$$(a) \quad L' \subseteq E_2$$

$$(b) \quad L'' = E'_1$$

$$(c) \quad L''' = E''_1$$

Thus, $H = E'_1 \oplus E''_1$ (if we choose $E'_1 \neq E''_1$), and $L' = E_2 \cap E'_1 \oplus E''_1$ (if we choose $E_2 \neq E'_1 \oplus E''_1$), so the configuration is determined uniquely. Hence $\mu = 1$.

Now we turn to $(a \circ a^2)_{0,1} = \lambda a + \mu b$. We have $\lambda = \langle A|A^2|B^2 \rangle_{0,1}$, which is zero by the proof of Proposition 4.

Next, we have $\mu = \langle A|A^2|A^2 \rangle_{0,1}$. As representatives of the homology classes A, A^2, A^2 , we try:

$$(a) \quad \{L \subseteq E_2\}$$

$$(b) \quad \{L = E'_1\}$$

$$(c) \quad \{L = E''_1\}$$

We will choose E_2, E'_1, E''_1 so that there are only finitely many configurations (K, V', V'', V''') — where V', V'', V''' are distinct two-dimensional subspaces containing the line K — such that

$$(a) \quad K \subseteq E_2$$

$$(b) \quad K = E'_1$$

$$(c) \quad K = E''_1$$

Thus, $E'_1 = E''_1$, but we may choose E_2, E'_1, E''_1 so that this is false. Hence $\mu = 0$.

Proposition 7. $b \circ b^2 = aq_2$.

Proof. Similar to the proof of Proposition 6.

This completes the computation of all quantum products of degree at most 6. We shall be able to evaluate the other quantum products without doing any further calculations of triple products.

Corollary 8. (i) $a \circ a^2 b = b^2 q_1 + q_1 q_2$. (ii) $b \circ a^2 b = a^2 q_2 + q_1 q_2$.

Proof. (i) $a \circ a^2 b = a \circ (a^2 \circ b) = (a \circ a^2) \circ b = (bq_1) \circ b = b \circ bq_1 = b^2 q_1 + q_1 q_2$. (ii) Similarly.

Corollary 9. (i) $a^2 \circ a^2 = b^2 q_1$. (ii) $b^2 \circ b^2 = a^2 q_2$. (iii) $a^2 \circ b^2 = q_1 q_2$.

Proof. (i) $a^2 \circ a^2 = (a \circ a - q_1) \circ a^2 = a \circ a \circ a^2 - a^2 q_1 = a \circ (bq_1) - a^2 q_1 = abq_1 - a^2 q_1 = b^2 q_1$. (ii), (iii)
Similarly.

Corollary 10. (i) $a^2 \circ a^2 b = aq_1 q_2$. (ii) $b^2 \circ a^2 b = bq_1 q_2$.

Proof. (i) $a^2 \circ a^2 b = a^2 \circ (a^2 \circ b) = (a^2 \circ a^2) \circ b = b^2 q_1 \circ b = (b^2 \circ b) q_1 = aq_1 q_2$. (ii) Similarly.

Corollary 11. $a^2 b \circ a^2 b = abq_1 q_2$.

Proof. $a^2 b \circ a^2 b = (a^2 \circ b) \circ a^2 b = a^2 \circ (a^2 q_2 + q_1 q_2) = b^2 q_1 q_2 + a^2 q_1 q_2 = abq_1 q_2$.

What are the relations defining the quantum cohomology algebra? By Theorem 2.2 of [Si-Ti], these are the “quantum versions” of the Borel relations

$$x_1 x_2 + x_2 x_3 + x_3 x_1 = 0, \quad x_1 x_2 x_3 = 0.$$

We obtain:

$$\begin{aligned} x_1 \circ x_2 + x_2 \circ x_3 + x_3 \circ x_1 &= a \circ (b - a) + (b - a) \circ (-b) + (-b) \circ a \\ &= -a \circ a - b \circ b + a \circ b \\ &= -q_1 - q_2 \end{aligned}$$

$$\begin{aligned} x_1 \circ x_2 \circ x_3 &= a \circ (b - a) \circ (-b) \\ &= -a \circ b \circ b + a \circ a \circ b \\ &= -ab \circ b + a \circ ab \\ &= -(a^2 + b^2) \circ b + a \circ (a^2 + b^2) \\ &= bq_1 - aq_2 \end{aligned}$$

Restricting attention to nonnegative powers of q_1, q_2 , we therefore have

$$\widetilde{QH}^*(F_3; \mathbf{C}) \cong \frac{\mathbf{C}[x_1, x_2, x_3, q_1, q_2]}{\langle x_1 x_2 + x_2 x_3 + x_3 x_1 + q_1 + q_2, x_1 x_2 x_3 + x_3 q_1 + x_1 q_2 \rangle}.$$

It should be noted that this information — the isomorphism type of $\widetilde{QH}^*(F_3; \mathbf{C})$ (or $QH^*(F_3; \mathbf{C})$) — does not determine all the quantum products.

APPENDIX 2: $QH^*(\Sigma_k)$

We shall investigate the Hirzebruch surfaces $\Sigma_k = \mathbf{P}(\mathcal{O}(0) \oplus \mathcal{O}(-k))$, where $\mathcal{O}(i)$ denotes the holomorphic line bundle on $\mathbf{C}P^1$ with first Chern class i . Since

$$\mathbf{P}(\mathcal{O}(0) \oplus \mathcal{O}(-k)) \cong \mathbf{P}(\mathcal{O}(k) \otimes (\mathcal{O}(0) \oplus \mathcal{O}(-k))) \cong \mathbf{P}(\mathcal{O}(k) \oplus \mathcal{O}(0))$$

(as complex manifolds), we have $\Sigma_k \cong \Sigma_{-k}$. Because of this, we shall assume from now on that $k \geq 0$. We shall use the following explicit description of Σ_k (see [Hi]):

$$\Sigma_k = \{([z_0; z_1; z_2], [w_1; w_2]) \in \mathbf{C}P^2 \times \mathbf{C}P^1 \mid z_1 w_1^k = z_2 w_2^k\}$$

The following four subvarieties of Σ_k will play an important role in our calculations.

$$\begin{aligned} X_1 &= \{z_2 = w_1 = 0\} \\ X_2 &= \{z_1 = z_2 = 0\} \\ X_3 &= \{z_1 = w_2 = 0\} \\ X_4 &= \{z_0 = 0\} \end{aligned}$$

If Σ_k is regarded as “ $\mathcal{O}(-k) \cup \infty$ -section”, then X_1 and X_3 are fibres, X_2 is the 0-section, and X_4 is the ∞ -section. We shall also denote by X_i the homology class in $H_2(\Sigma_k; \mathbf{Z})$ represented by the variety X_i . As usual, the Poincaré dual cohomology class in $H^2(\Sigma_k; \mathbf{Z})$ will be denoted x_i .

Proposition 1. *The relations between the cohomology classes x_1, x_2, x_3, x_4 are:*

- (1) $x_1 = x_3, x_4 = x_2 + kx_1$.
- (2) $x_1 x_3 = x_2 x_4 = 0, x_1 x_2 = x_1 x_4 = x_2 x_3 = x_3 x_4 = z$, where z is a generator of $H^4(\Sigma_k; \mathbf{Z}) \cong \mathbf{Z}$.
- (3) $x_1^2 (= x_3^2) = 0, x_2^2 = -kz, x_4^2 = kz$.

Proof. (1) follows from consideration of the meromorphic functions on Σ_k defined by the expressions $w_1/w_2, z_1/z_0$. (2) follows from consideration of intersections. (3) is a consequence of (1) and (2).

The cohomology algebra of Σ_k is generated by x_1 and x_2 , by the Leray-Hirsch Theorem (Theorem 5.11 and (20.7) of [Bo-Tu]). From this, and by using the proposition, we obtain:

$$\begin{aligned} H^*(\Sigma_k; \mathbf{Z}) &\cong \mathbf{Z}[x_1, x_2, x_3, x_4]/\langle x_1 - x_3, x_4 - x_2 - kx_1 \rangle + \langle x_1 x_3, x_2 x_4 \rangle \\ &\cong \mathbf{Z}[x_1, x_4]/\langle x_1^2, x_4^2 - kz \rangle. \end{aligned}$$

The particular form of the first description — where we have written the ideal of relations as a sum of two separate ideals — is chosen to be consistent with the usual description of the cohomology algebra of a toric variety (see [Od]).

It can be verified (from this description) that⁵ $H^*(\Sigma_k; \mathbf{Z}) \cong H^*(\Sigma_{k+2}; \mathbf{Z})$, and that $H^*(\Sigma_k; \mathbf{Z}) \not\cong H^*(\Sigma_{k+1}; \mathbf{Z})$. This is consistent with the results of [Hi], where it was shown that Σ_k is homeomorphic to Σ_l if and only if $k - l$ is even.

⁵A suitable isomorphism is given by $x_2 \mapsto x_2 + x_1, x_4 \mapsto x_4 - x_1$.

We may choose additive generators of $H^*(\Sigma_k; \mathbf{Z})$ as follows:

$$\begin{array}{ccc} H^0(\Sigma_k; \mathbf{Z}) & H^2(\Sigma_k; \mathbf{Z}) & H^4(\Sigma_k; \mathbf{Z}) \\ 1 & x_1 & z = x_1 x_4 \\ & x_4 & \end{array}$$

From the proposition we have:

$$x_1^c = x_2 = x_4 - kx_1, \quad x_4^c = x_3 = x_1.$$

We shall make use of the following representatives of homology classes of Σ_k . These are analogous to the Schubert varieties in flag manifolds and Grassmannians, but the situation is more rigid now:

- X_1 : represented by X_1, X_3 or any “fibre”
- X_2 : represented only by X_2
- X_4 : represented only by X_4

We turn now to holomorphic maps $f : \mathbf{CP}^1 \rightarrow \Sigma_k$. Such maps have a very explicit description in terms of polynomials. Namely, f is of the form

$$f = ([p_4; p_2 p_3^k; p_2 p_1^k], [p_1; p_3])$$

where p_1, p_2, p_3, p_4 are arbitrary complex polynomials such that p_1, p_3 have no common factor, and p_2, p_4 have no common factor. (The notation is chosen so that $p_i \equiv 0$ if and only if $f(\mathbf{CP}^1) \subseteq X_i$.)

We define the “degree” of f by $\deg f = (d, e)$, where

$$\begin{aligned} d &= \max\{\deg p_4, \deg p_2 p_3^k, \deg p_2 p_1^k\} \\ e &= \max\{\deg p_1, \deg p_3\} \end{aligned}$$

Proposition 2.

(1) $\deg f = (d, e)$ if and only if $[f] = dX_1 + eX_2$ ($\in \pi_2(\Sigma_k) \cong H_2(\Sigma_k; \mathbf{Z})$).

(2) There exists a holomorphic map f such that $\deg f = (d, e)$ if and only if either

(a) $d \geq ke \geq 0$, or

(b) $d = 0$, $e > 0$.

Proof. (1) Let H_1, H_2 be the restrictions to Σ_k of the tautologous line bundles on $\mathbf{CP}^1, \mathbf{CP}^2$. Then the condition $\deg f = (d, e)$ means that $d = -c_1 f^* H_2$ and $e = -c_1 f^* H_1$. It may be verified that $c_1 H_1 = -x_1$ and $c_1 H_2 = -x_4$ (by using the fact that the first Chern class of the dual of the tautologous line bundle on \mathbf{CP}^n is Poincaré dual to a hyperplane in \mathbf{CP}^n). Hence the condition $\deg f = (d, e)$ means that $d = f^* x_4$ and $e = f^* x_1$. Let us write $[f] = uX_1 + vX_2$. Then we have

$$u = \langle x_4, uX_1 + vX_2 \rangle = \langle x_4, f_*[\mathbf{CP}^1] \rangle = \langle f^* x_4, [\mathbf{CP}^1] \rangle = d$$

and similarly $v = e$. This completes the proof of (1).

(2) This is obvious from the polynomial expression for f .

We need one more ingredient in order to compute the quantum cohomology of Σ_k :

Proposition 3. $c_1 T\Sigma_k = 2x_4 - (k-2)x_1$.

Proof. See section 3.3 of [Od], for example.

Let us choose X_1, X_2 as a basis of $H_2(\Sigma_k; \mathbf{Z})$, and let us denote the corresponding quantum parameters by q_1, q_2 . By the usual definition we have

$$\begin{aligned} |q_1| &= 2\langle c_1 T\Sigma_k, X_1 \rangle = 2\langle 2x_4 - (k-2)x_1, X_1 \rangle = 4 \\ |q_2| &= 2\langle c_1 T\Sigma_k, X_2 \rangle = 2\langle 2x_4 - (k-2)x_1, X_2 \rangle = 2(2-k) \end{aligned}$$

Warning: The usual definition of $|q_1|$ and $|q_2|$ is designed to indicate the “expected” dimension of $\text{Hol}_{d,e}(\mathbf{CP}^1, \Sigma_k)$. In case (a), i.e. $d \geq ke \geq 0$, this is correct:

$$\begin{aligned} \dim_{\mathbf{C}} \text{Hol}_{d,e}(\mathbf{CP}^1, \Sigma_k) &= 2d + (2-k)e + 2 \quad (\text{from the polynomial description}) \\ \dim_{\mathbf{C}} \Sigma_k + \langle c_1 T\Sigma_k, dX_1 + eX_2 \rangle &= 2d + (2-k)e + 2. \end{aligned}$$

In case (b), however, it is not correct in general. Here we have $d = 0$ and $e > 0$, so — if $k > 0$ — the image of f must be contained in X_2 . We obtain:

$$\begin{aligned} \dim_{\mathbf{C}} \text{Hol}_{d,e}(\mathbf{CP}^1, \Sigma_k) &= 2e + 1 \quad (\text{from the polynomial description}) \\ \dim_{\mathbf{C}} \Sigma_k + \langle c_1 T\Sigma_k, dX_1 + eX_2 \rangle &= (2-k)e + 2. \end{aligned}$$

This means that we cannot rely on the “numerical condition” when performing our calculations. More precisely, the problem is that Σ_k is Fano only for $k = 0, 1$ and convex only for $k = 0$ — see the remarks at the end of §3 and at the end of this section. Because of this, our calculations from now on will be merely heuristic. However, we shall arrive at the correct answer, and some references for a rigorous approach will be given later.

In calculating the triple products $\langle A|B|C \rangle_D$, we shall only need the cases $D = (0,0)$, $(1,0)$, or $(0,1)$; this follows from the polynomial representation of holomorphic maps. The polynomial representation shows also that

(1) the holomorphic maps of degree $(1,0)$ are precisely the “fibres”, and

(2) the holomorphic maps of degree $(0,1)$ are given by the “0-section”, X_2 .

To calculate $\langle A|B|C \rangle_D$, we need to find representatives of the homology classes A, B, C such that there is only a finite number of configurations $(P_1, P_2, P_3, \mathbf{CP}^1)$ such that $P_1 \in A$, $P_2 \in B$, $P_3 \in C$,

where \mathbf{CP}^1 represents the homotopy class D and P_1, P_2, P_3 are three distinct points in this \mathbf{CP}^1 . If such representatives do not exist, then we shall say that $\langle A|B|C \rangle_D$ is “not computable”. If, for fixed A and B , the necessary triple products $\langle A|B|C \rangle_D$ are not computable, we shall say that the quantum product $a \circ b$ is “not directly computable”. (It may still be possible to compute $a \circ b$ indirectly, by expressing it in terms of other known quantum products — or, of course, by using more general representatives of homology classes than we have so far allowed ourselves.)

Calculations for $\Sigma_0 = \mathbf{CP}^1 \times \mathbf{CP}^1$

In this case, we can rely on the numerical condition. We have $|q_1| = |q_2| = 4$, and $x_1 = x_3, x_2 = x_4$. We obtain

$$\begin{aligned} x_1 \circ x_1 &= q_2 \\ x_4 \circ x_4 &= q_1 \\ x_1 \circ x_4 &= x_1 x_4. \end{aligned}$$

The quantum cohomology algebra is

$$\widetilde{QH}^*(\Sigma_0; \mathbf{C}) \cong \frac{\mathbf{C}[x_1, x_4, q_1, q_2]}{\langle x_1^2 - q_2, x_4^2 - q_1 \rangle}.$$

This is just the tensor product of two copies of the quantum cohomology algebra of \mathbf{CP}^1 .

Calculations for Σ_1

In this case too, we can rely on the numerical condition. The problematical case is $D = (0, 1)$, i.e. $d = 0, e = 1$, but in fact the space of holomorphic maps of degree D has the correct dimension, as $2e + 1 = e + 2$ in this case.

We take $|q_1| = 4$ and $|q_2| = 2$, and proceed in the usual way. We shall try to calculate the six possible quantum products involving the cohomology classes $x_1 (= x_3), x_2, x_4$.

Proposition 4. $x_1 \circ x_1 = x_2 q_2$.

Proof. Let us write

$$x_1 \circ x_1 = x_1^2 + \lambda q_1 + \mu q_2$$

where $\lambda \in H^0(\Sigma_1; \mathbf{Z})$ and $\mu \in H^2(\Sigma_1; \mathbf{Z})$.

We have $\lambda = \langle X_1 | X_1 | Z \rangle_{1,0}$, where Z is the generator of $H_0(\Sigma_1; \mathbf{Z})$. Let us choose X_1, X_3 , and any point of Σ_1 as representatives of the three homology classes. Then each holomorphic map of degree $(1, 0)$ (i.e. each “fibre”) fails to intersect all three representatives. Hence $\lambda = 0$.

To calculate μ , we must calculate $\langle \mu, Y \rangle = \langle X_1 | X_1 | Y \rangle_{0,1}$, for two independent homology classes $Y \in H_2(\Sigma_1; \mathbf{Z})$.

The triple product $\langle X_1 | X_1 | X_1 \rangle_{0,1}$ is equal to 1, because we may choose three distinct “fibres” representing X_1 , and then there is a unique configuration (P_1, P_2, P_3, X_2) which intersects these fibres (respectively) in the points P_1, P_2, P_3 .

The triple product $\langle X_1|X_1|X_2 \rangle_{0,1}$ is not computable, because there are infinitely many configurations (P_1, P_2, P_3, X_2) with the appropriate intersection property, whatever representatives of X_1, X_1, X_2 are chosen.

The triple product $\langle X_1|X_1|X_4 \rangle_{0,1}$ is equal to zero, because $X_2 \cap X_4 = \emptyset$.

We conclude that $\langle \mu, X_1 \rangle = 1$ and $\langle \mu, X_4 \rangle = 0$, and hence $\mu = x_2$.

Proposition 5. $x_2 \circ x_2$ is not directly computable.

Proof. (There is only one representative of the homology class X_2 .)

Proposition 6. $x_4 \circ x_4$ is not directly computable.

Proof. (There is only one representative of the homology class X_4 .)

Proposition 7. $x_1 \circ x_4 = x_1 x_4$.

Proof. Let us write $x_1 \circ x_4 = x_1 x_4 + \lambda q_1 + \mu q_2$. We have $\lambda = \langle X_1|X_4|Z \rangle_{1,0}$. By taking Z as any point in the complement of $X_1 \cup X_4$, we see that $\lambda = 0$. A holomorphic map of degree $(0, 1)$ cannot intersect X_4 , so $\mu = 0$.

Proposition 8. $x_1 \circ x_2$ is not directly computable.

Proof. (The triple products $\langle X_1|X_2|Y \rangle_{0,1}$ are not computable, for any Y .)

Proposition 9. $x_2 \circ x_4 = q_1$.

Proof. Let us write $x_2 \circ x_4 = x_2 x_4 + \lambda q_1 + \mu q_2$. We have $\lambda = \langle X_2|X_4|Z \rangle_{1,0}$, and this is equal to 1, as there is a unique ‘‘fibre’’ which intersects X_2, X_4 and a point in the complement of $X_2 \cup X_4$. The triple products $\langle X_2|X_4|Y \rangle_{0,1}$ are necessarily zero, so we have $\mu = 0$.

To summarize, we have now computed the following quantum products for Σ_1 :

$$\begin{aligned} x_1 \circ x_1 &= x_2 q_2 \\ x_1 \circ x_4 &= z \\ x_2 \circ x_4 &= q_1. \end{aligned}$$

From the relation $x_4 = x_2 + x_1$, we obtain the three quantum products which were not directly computable:

$$\begin{aligned} x_4 \circ x_4 &= z + q_1 \\ x_1 \circ x_2 &= z - x_2 q_2 \\ x_2 \circ x_2 &= -z + q_1 + x_2 q_2. \end{aligned}$$

The quantum cohomology algebra is therefore

$$\widetilde{QH}^*(\Sigma_1; \mathbf{C}) \cong \frac{\mathbf{C}[x_1, x_4, q_1, q_2]}{\langle x_1^2 - x_2 q_2, x_4^2 - z - q_1 \rangle}.$$

Hirzebruch surfaces are examples of toric manifolds, and it is possible to describe rational curves in general toric manifolds quite explicitly (see [Gu1]). A variant of quantum cohomology of toric manifolds was introduced in [Ba], and its relation with the usual quantum cohomology algebra was discussed in [Co-Ka] (examples 8.1.2.2 and 11.2.5.2) and in [Si] (section 4). Unfortunately the situation is quite complicated because toric manifolds are not convex in general (cf. the remarks at the end of §3). We have already seen this in the case of the Hirzebruch surface Σ_k , which is convex if and only if $k = 0$, i.e. if and only if Σ_k is homogeneous. An alternative way of proceeding in the non-convex case is given in section 15 of [Au2]; the calculations for Σ_1 (or rather, for the isomorphic space Σ_{-1}) are given in section 16 of that article. A rigorous treatment of the quantum cohomology of Σ_1 — and more generally, of the manifold obtained by blowing up \mathbf{CP}^n at a point — appears in [Ga].

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